

MTH512-Course Portfolio-Fall 2019

Ayman Badawi

Table of contents

1 Section 1: Syllabus

COURSE SYLLABUS

A	Course Title & Number	ADVANCED LINEAR ALGEBRA: MTH 512
B	Pre/Co-requisite(s)	Admission to MSMTH program
C	Number of credits	3
D	Faculty Name	Ayman Badawi
E	Term/ Year	Fall 2019
G	Instructor Information	see below

Instructor	Office	Telephone	Email
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Office Hours: By appointment

H Course Description from Catalog

Topics include the proof-based theory of matrices, determinants, vector spaces, linear spaces, linear transformations and their matrix representations, linear systems, linear operators, eigenvalues and eigenvectors, invariant subspaces of operators, spectral decompositions, functions of operators, and applications to science, industry, and business.

I Course Learning Outcomes

Upon completion of the course, students will be able to:

1. Write proofs for simple questions.
2. Demonstrate an understanding of vector spaces, subspaces and change of basis.
3. Solve and analyze matrices using eigenvalues and eigenvectors.
4. Demonstrate an understanding of canonical forms and Jordan forms.
5. Demonstrate an understanding of inner-product spaces, norms, orthonormal bases, operators on inner-product space.
6. Demonstrate an understanding of spectral theory, singular value decomposition and applications of linear algebra.
7. Apply skills learned in linear algebra, for example Least Square Method.

J Textbook and other Instructional Material and Resources

MAIN: Class notes. Materials on I-learn and my personal webpage http://ayman-badawi.com/MTH%20512.html

Secondary: Sheldon Axler, *Linear Algebra Done Right*, 1997(*any Edition will do)*. The book is available on the web as free download. Any E-text book treats the above concepts will do.

K Teaching and Learning Methodologies

The teaching and learning tools used in this course to deliver the subject matter include black board with chocks (if available) but the current white board and markers will do, formal lectures, class discussions.

L Grading Scale, Grading Distribution, and Due Dates

Grading Scale

Excellent	
A	Equals 4.00 grade points
Meet Expectation	
A-	Equals 3.80 grade points
B+	Equals 3.30 grade points
B	Equals 3.00 grade points

Below Expectation	
B-	Equals 2.70 grade points
C+	Equals 2.30 grade point
C	Equals 2.00 grade point
Fail	
F	Equals 0.00 grade points
Academic Integrity Violation Fail	
XF	Equals 0.00 grade points
Withdrawal Fail	
WF	Equals 0.00 grade points

Grading Distribution

Assessment	Weight	Date
Homework	15 %	
Exam 1	25 %	
Exam 2	25 %	
Final Exam	35 %	
Total	100 %	

M	**Explanation of Assessments**	Exams, homework assignments will include simple proofs. So students are expected to master some of the techniques that are commonly used in linear algebra.
N	**Student Academic Integrity Code Statement**	Student must adhere to the Academic Integrity code stated in the graduate catalog.

SCHEDULE (BUT NOT IN ORDER)

No addendum, make-up exams, or extra assignments to improve grades will be given.

#	WEEK	CHAPTER/SECTIONS	NOTES
1	1	**Vector Spaces**	Definition Examples
2	2	**Subspaces and Direct Sums**	Definition Examples Proofs of some simple results

4	2	**Span, Linear Independence, Bases, Dimension , and Linear Transformation**	Examples Proofs of some simple results
6	1	**Exam 1**	
7	2	**Eigenvalues, Eigenvectors, and Invariant Subspaces on Real Vector Spaces**	Examples Using the methods in analyzing some basic facts on matrices
9	2	**Inner Products, Orthonormal Bases, Orthogonal Projections and Minimization Problems (Least Square Method)**	Definition Examples Simple proofs Application
11	1	**Operators on Inner-Product Spaces**	Examples Simple and Basic Proofs
12	1	**Exam 2**	Exam 2 : Covers all materials after Exam 1
13	1	**The Characteristic polynomial and the minimal polynomial of an operator, and its decomposition**	Examples Simple Proofs
14	1	**Canonical forms, Rational and Jordan Forms**	Definition Examples

15	1	**Spectral theory, Singular Value Decomposition**	Examples
16	1	**Review before a comprehensive final exam**	

2 Section 3: Handouts and other Materials

2.1 Reviews for Exam One

Review Exam one MTH 512 , Fall 2019

Ayman Badawi

REMARK 1. You should know the following concepts

(i) Orthogonal, Orthonormal and how to make orthogonal basis an orthonormal basis.

(ii) Solving system of linear equations (in particular homogeneous system) and write the solution set as span of orthogonal (orthonormal) basis.

(iii) The meaning of Independent number (dimension) and how to find this number if a subspace is given.

(iv) If Q lives in span of independent points (say $Q_1, Q_2, ..., Q_k$), then there exist UNIQUE real numbers $a_1, ..., a_k$ such that $Q = a_1Q_1 + ... + a_kQ_k$

(v) nonzero Orthogonal points imply independent but not vice-versa.

(vi) $CD = L$ (say C is $n \times k$ and D is $k \times m$). Then each column of L is a linear combination of Columns of C. Let $C_1, ..., C_k$ be the columns of C. Then for example, the fourth column of L, $L_4 = d_{1,4}C_1 + d_{2,4}C_2 + ... + d_{k,4}C_k$ (where $d_{1,4}, ..., d_{k,4}$ are the numbers in the fourth column of D.

(vii) You should be aware of the METHOD that I discussed in class, how to check if $Q_1, ..., Q_n$ are independent or not

(viii) $Rank(A) + Nullity(A)$ = number of columns of A [note $Nullity(A)$ = IN(Solution set of the homogeneous system $AX = O$) = number of free variables]

(ix) Show that a subset of R^n is a subspace by writing the set as Span of some points (and then a span of independent points).

(x) A subset $D = \{(\ \ , \ \ , ..., \ \ )|a, b, c, d... \in R\}$ of R^n is a subspace IFF D can be rewritten so that each coordinate is a linear combination of the linear variables a, b, c, d, ... (see class notes)

(xi) Let say A is $n \times n$. Is 4 an eigenvalue of A? It might be difficult to find the roots of $C_A(\alpha)$. Hence an easy way to answer the question is to find Rank $(4I_n - A))$. If the Rank = n, the answer is no (hence A is invertible). If the Rank is < n, then the answer is yes.

(xii) Let say A is $n \times n$. Is 4 an eigenvalue of A?If yes, then find E_4. It might be difficult to find the roots of $C_A(\alpha)$. Hence an easy way to answer the question (note that here you need to find E_4) is to find the solution set of the homogeneous system $(4I_n - A)X = 0$.

(xiii) Understand the meaning of eigenvalue, eigenvector (eigen-point).

(xiv) A $(k \times m)$ is row equivalent to B (assume 7 row operation applied on A in order to get B). You should know how to go back from B to A (see class notes). You should be able to find 7 elementary matrices (each is of size $k \times k$), say $E_1, ..., E_7$ such that $E_1E_2 \cdots E_7A = B$. Also you should know how to find 7 elementary matrices $F_1, ..., F_7$ (again each is $k \times k$) such that $F_1F_2 \cdots F_7B = A$.

(xv) Meaning of diagnolizable over R and how to find D and Q. (see Class Notes)

(xvi) How to check if A is diagnolizable over R or not (see class notes, big Theorem).

(xvii) how to calculate determinant using ROW-Operations.

(xviii) $C_A(\alpha) = |\alpha I_n - A|$ (note other books they use $|A - \alpha I_n|$). Using our notation, $|A|$ is (plus or minus) the constant-term of $C_A(\alpha)$. Trace of (A) ALWAYS equal - (coefficent of x) in $C_A(\alpha)$. (I think I told you I am not sure if I need minus, now I confirm yes it is always minus). For example if $C_A(\alpha) = \alpha^3 + 7\alpha - 22$. Then $|A| = (plus, minus)22$, but Trace(A) = - 7.

(xix) Let α be a real number, A be $n \times n$. Then

a. $|\alpha A| = \alpha^n|A|$.

b. If A is invertible (nonsingular), then $|A^{-1}| = 1/|A|$.

c. If A is similar to B (i.e., $A = DBD^{-1}$), then $|A| = |B|$, and $C_A(\alpha) = C_B(\alpha)$

d. If A is invertible and a is an eigenvalue of A, then $1/a$ is an eigenvalue of A^{-1} (easy proof)

e. If a is an eigenvalue of A, then a^k is an eigenvalue of A^k.

f. $|A + B|$ NEED NOT EQUAL $|A| + |B|$ (you can find an example easily)

g. $|A| = |A^T|$ and $C_A(\alpha) = C_{A^T}(\alpha)$.

(xx) If A is invertible, you need to know how to find A^{-1} using the METHOD $[A|I_n]$ ROW-OPERATIONS $[I_n|A^{-1}]$. Recall if A is 2×2 and invertible, then it is easy to find A^{-1}, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $|A| = ad - bc$. If $|A| \neq 0$, then $A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

(xxi) $Rank(A) = Rank(A^T)$

(xxii) If A is row-equivalent to B, then Rank(A) = Rank (B) (easy)

(xxiii) If A has exactly k independent rows, then A has exactly k independent columns.

(xxiv) Assume A is 3×4. Assuming A is row equivalent to $B = \begin{bmatrix} 2 & 0 & 4 & 3 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Then

a. Quickly, Rank(A) = Rank(B) = 2.

b. Recall $Row(A)$ = span of rows of A and $Row(A) = span\{R_1, R_2\} = span\{(2,0,4,3),(0,0,1,7)\}$. What does that mean? EACH ROW OF A is a linear combination of (2, 0, 4, 3) and (0, 0, 1, 7) (nice meaning!)

c. Recall Col(A) = Span columns of A. Recall how to find basis to the column space of A. Stare at B, locate the columns in B that have "leaders". Here , we have B_1 and B_3. A basis for Col(A) must be chosen from A and not from B (why? because we are using ROW-operations on A (not Column operations), so we cannot gurantee that the column of B "live" inside Col(A)). Since the leaders in B are located in B_1, B_3, we choose A_1, A_3 to form a basis for Col(A). Hence $Col(A) = Span\{A_1, A_3\}$. Again, what does that mean? Each column of A is a linear combination of A_1 and A_3.

(xxv) Let $B = \{D = (2,0,3), T = (0,-1,2), L = (0,0,1)\}$ be a basis for R^3 and $F = (4,5,9) \in R^3$. Find $[F]_B$ and explain the meaning of your answer. We know that $[F]_B = Q^{-1}F^T$ (see class notes), where Q is an invertible 3×3 matrix, First Column of Q (Q_1) is the point D^T, Q_2 is the point T^T and Q_3 is the point L^T. Now enjoy the calculation. Assume the answer is $[F]_B = (c_1, c_2, c_3)$. This means that $F = (4,5,9) = c_1D + c_2T + c_3L$.

(xxvi) If you need to check your calculation, I recommend the following online Calculators:

(1) Linear Algebra Tool Kit (Strongly RECOMMENDED)

(2) GRAM-SCHMIDT CALCULATOR

(3) CHARACTERISTIC POLYNOMIAL CALCULATOR

(4) EIGENVALUE AND EIGENVECTOR CALCULATOR

(5) DIAGONALIZE MATRIX CALCULATOR

(I will add these LINKS soon in Lectur/Notes Folder on I-Learn)

Faculty information

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2.2 HW I

Name——————————————, ID ————————

Assignment I MTH 512 , Fall 2019

Ayman Badawi

QUESTION 1. Let Q_1, Q_2, Q_3 be independent points in R^n such that $span\{Q_1, Q_2, Q_3\} \neq R^n$.

(i) What is the smallest n so that $Q_1, Q_2, Q_3 \in R^n$?

(ii) Prove that $Q_1 + Q_2, Q_1 + Q_3, Q_2 + Q_3$ are independent points in R^n.

(iii) Assume that Q_1, Q_2, Q_3 are orthogonal and $L = span\{Q_1, Q_2, Q_3\}$. Given $Q \in L$. Hence $Q = a_1Q_1 + a_2Q_2 + a_3Q_3$ for some real numbers a_1, a_2, a_3. Prove that $a_1 = \frac{Q \cdot Q_1}{||Q_1||^2}$, $a_2 = \frac{Q \cdot Q_2}{||Q_2||^2}$, $a_3 = \frac{Q \cdot Q_3}{||Q_3||^2}$.

QUESTION 2. Let $D = span\{(2a + 3, -b + 1, 6a - 2b + 11, 0) \mid a, b \in R\}$

(i) Convince me that D is a subspace of R^4. (I guess, it is enough to rewrite D as span)

(ii) Find an orthogonal basis for D.

QUESTION 3. Let $A = \begin{bmatrix} 5 & 3 & 1 & 1 \\ 1 & 3 & -1 & 0 \\ 2 & -6 & 4 & 1 \\ 4 & -12 & -4 & 1 \end{bmatrix}$. Is 6 an eigenvalue of A? If yes, then find E_6 and find an orthogonal basis for E_6.

QUESTION 4. Let A be an $n \times n$ matrix and r be a fixed real number. Suppose that the sum of all numbers (entries) of each row of A equals to r. Prove that r is an eginvalue of A.

QUESTION 5. Given A is a 4×4 matrix such that $A \ \overbrace{3R_2} \ B \ \overbrace{-6R_1 + R_4 \to R_4} \ C \ \overbrace{R_3 \leftrightarrow R_2} \ D \ \overbrace{-2R_2} \ F = \begin{bmatrix} 0 & 0 & 4 & 6 \\ 1 & 3 & -1 & 0 \\ 0 & -6 & 4 & 1 \\ 4 & 12 & -4 & 2 \end{bmatrix}$. Find $|A|$, $|C|$, and $|D|$.

QUESTION 6. (i) Convince me that $L = \{(a, b^3, 0) \mid a, b \in R\}$ is a subspace of R^3.

(ii) Convince me that $L = \{(a, 0, b^2) \mid a, b \in R\}$ is not a subspace of R^3.

(iii) Convince me that $L = \{(b, b^3, 0) \mid b \in R\}$ is not a subspace of R^3.

(iv) Convince me that 3 is not an eiginvalue of $A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

(v) Let A be a 4×4 matrix such that A_2 (second column of A) is identical to A_4 (4th column of A). Consider the following system of L. E. $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = A_2$. Convince me that the system has infinitely many solutions. Give me 3 distinct points that belong to the solution set of the system.

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2.3 Solution to HW I

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Assignment I: MTH 512

47/50

* Question 1.

i) The smallest n so that $Q_1, Q_2, Q_3 \in \mathbb{R}^n$ is 4

since if we take $n=3$ and $Q_1 = (1,0,0)$, $Q_2 = (0,1,0)$, $Q_3 = (0,0,1)$ $\in \mathbb{R}^3$

then $\text{span}\{Q_1, Q_2, Q_3\} = \mathbb{R}^3$ (contradiction).

4/4

ii) To prove that Q_1+Q_2, Q_1+Q_3, Q_2+Q_3 are independent in $\mathbb{R}^n$ we need to prove that the only solution of:

$a(Q_1+Q_2) + b(Q_1+Q_3) + c(Q_2+Q_3) = 0$ is $a=b=c=0$

so $aQ_1 + aQ_2 + bQ_1 + bQ_3 + cQ_2 + cQ_3 = 0$

$\Rightarrow (a+b)Q_1 + (a+c)Q_2 + (b+c)Q_3 = 0$

since Q_1, Q_2 and Q_3 are independent in $\mathbb{R}^n$ then

$a+b=0$, $a+c=0$, and $b+c=0$

$\Rightarrow \left.\begin{matrix} b=-a \\ c=-a \end{matrix}\right\} -a-a=0 \Rightarrow -2a=0 \Rightarrow a=b=c=0$

4/4

thus (Q_1+Q_2), (Q_1+Q_3), (Q_2+Q_3) are independent in $\mathbb{R}^n$

iii) we know that Q_1, Q_2 and Q_3 are orthogonal

and $Q = a_1 Q_1 + a_2 Q_2 + a_3 Q_3$

- for a_1: $Q.Q_1 = a_1 Q_1 Q_1 + a_2 Q_2 Q_1 + a_3 Q_3 Q_1$

$$\Rightarrow Q.Q_1 = a_1 \|Q_1\|^2 + 0 + 0$$ (because they are orthogonal)

$$\Rightarrow a_1 = \frac{Q.Q_1}{\|Q_1\|^2}$$

- for a_2: $Q.Q_2 = a_1 Q_1 Q_2 + a_2 Q_2 Q_2 + a_3 Q_3 Q_2$

$$Q.Q_2 = 0 + a_2 \|Q_2\|^2 + 0$$

$$\Rightarrow a_2 = \frac{Q.Q_2}{\|Q_2\|^2}$$

- for a_3: $Q.Q_3 = a_1 Q_1 Q_3 + a_2 Q_2 Q_3 + a_3 Q_3 Q_3$

$$Q.Q_3 = 0 + 0 + a_3 \|Q_3\|^2$$

$$\Rightarrow a_3 = \frac{Q.Q_3}{\|Q_3\|^2}$$

5/5

* Question 2: $D = \text{span}\{(2a+3, -b+1, 6a-2b+11, 0) \mid a,b \in \mathbb{R}\}$

i) $D = \{(2a+3, -b+1, 6a+9-2b+2, 0) \mid a,b \in \mathbb{R}\}$

$= \{(2a+3)(1,0,3,0) + (-b+1)(0,1,2,0) \mid a,b \in \mathbb{R}\}$

$= \text{span}\{(1,0,3,0), (0,1,2,0)\}$

thus D is a subspace of $\mathbb{R}^4$ because it can be written as span.

2/2

ii) let $Q_1 = (1,0,3,0)$ and $Q_2 = (0,1,2,0)$

$\{w_1, w_2\}$ is the orthogonal basis of D where:

- $w_1 = Q_1 = (1,0,3,0)$
- $w_2 = Q_2 - \dfrac{Q_2 \cdot w_1}{\|w_1\|^2} w_1$

$= (0,1,2,0) - \dfrac{6}{10}(1,0,3,0)$

$= (0,1,2,0) - \left(\frac{3}{5}, 0, \frac{9}{5}, 0\right)$

$= \left(-\frac{3}{5}, 1, \frac{1}{5}, 0\right)$

thus $\{(1,0,3,0), (-\frac{3}{5}, 1, \frac{1}{5}, 0)\}$ is the orthogonal basis of D.

3/3

* Question 3: $A = \begin{bmatrix} 5 & 3 & 1 & 1 \\ 1 & 3 & -1 & 0 \\ 2 & -6 & 4 & 1 \\ 4 & -12 & -4 & 1 \end{bmatrix}$

To see if 6 is an eigenvalue of A, we need to show that $\{(0,0,0,0)\}$ is not the solution set of: $(6I_4 - A)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

• $\begin{pmatrix} 1 & -3 & -1 & -1 \\ -1 & 3 & 1 & 0 \\ -2 & 6 & 2 & -1 \\ -4 & 12 & 4 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 & \\ \end{array}$$
$$\left(\begin{array}{cccc|c} 1 & -3 & -1 & -1 & 0 \\ -1 & 3 & 1 & 0 & 0 \\ -2 & 6 & 2 & -1 & 0 \\ -4 & 12 & 4 & 5 & 0 \end{array}\right) \xrightarrow[4R_1 + R_4 \to R_4]{R_1 + R_2 \to R_2,\ 2R_1 + R_3 \to R_3} \left(\begin{array}{cccc|c} 1 & -3 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

$$\Rightarrow \begin{cases} x_4 = 0 \\ x_1 - 3x_2 - x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_4 = 0 \\ x_1 = 3x_2 + x_3 \end{cases} \xrightarrow{\text{take } x_2 = a \text{ and } x_3 = b} \begin{cases} x_1 = 3a + b \\ x_2 = a \\ x_3 = b \\ x_4 = 0 \end{cases}$$

so the solution set $= \{(3a+b), a, b, 0 \mid a, b \in \mathbb{R}\}$

$= \cancel{\text{span}}\ \{a(3,1,0,0) + b(1,0,1,0) \mid a,b \in \mathbb{R}\}$

so $E_6 = \text{Span}\{(3,1,0,0), (1,0,1,0)\}$

thus 6 is an eigenvalue of A.

5/5

. Question 3:

Let $Q_1 = (3,1,0,0)$ and $Q_2 = (1,0,1,0)$

$\{w_1, w_2\}$ is the orthogonal basis for E_6 where

. $w_1 = Q_1 = (3,1,0,0)$

. $w_2 = Q_2 - \dfrac{Q_2 \cdot w_1}{\|w_1\|^2} w_1$

$= (1,0,1,0) - \dfrac{3}{10}(3,1,0,0)$

$= \left(\dfrac{1}{10}, -\dfrac{3}{10}, 1, 0\right)$

thus $\{(3,1,0,0), \left(\frac{1}{10}, -\frac{3}{10}, 1, 0\right)\}$ is the orthogonal basis for E_6.

* Question 4: Let A be $n \times n$ matrix and r a fixed real number where ~~all~~ the sum of all numbers of each row of A is equal to r.

Now consider the non-zero point $Q = (1, 1, \dots, 1) \in \mathbb{R}^n$ (all entries of Q is 1)

$$A Q^T = \underset{(n \times n)}{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}} \underset{(n \times 1)}{\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}} = \begin{pmatrix} a_{11} + a_{12} + \cdots + a_{1n} \\ a_{21} + a_{22} + \cdots + a_{2n} \\ \vdots \\ a_{n1} + a_{n2} + \cdots + a_{nn} \end{pmatrix} = \underset{(n \times 1)}{\begin{pmatrix} r \\ r \\ \vdots \\ r \end{pmatrix}}$$

and $$r Q^T = r \underset{\cancel{(n\times 1)}}{\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}} = \begin{pmatrix} r \\ r \\ \vdots \\ r \end{pmatrix}$$

thus $A Q^T = r Q^T$

So r is an eigen value of A because there exists a non-zero point in $\mathbb{R}^n$, Q, such that $A Q^T = r Q^T$

5/5

* Question 5: $F = \begin{bmatrix} 0 & 0 & 4 & 6 \\ 1 & 3 & -1 & 0 \\ 0 & -6 & 4 & 1 \\ 4 & 12 & -4 & 2 \end{bmatrix}$

$$|F| = (-1)^{1+3} \cdot 4 \begin{vmatrix} 1 & 3 & 0 \\ 0 & -6 & 1 \\ 4 & 12 & 2 \end{vmatrix} + (-1)^{1+4} \cdot 6 \begin{vmatrix} 1 & 3 & -1 \\ 0 & -6 & 4 \\ 4 & 12 & -4 \end{vmatrix}$$

$$= 4 \left[(-1)^{1+1} \cdot 1 \begin{vmatrix} -6 & 1 \\ 12 & 2 \end{vmatrix} + (-1)^{1+2} \cdot 3 \begin{vmatrix} 0 & 1 \\ 4 & 2 \end{vmatrix} \right] - 6 \left[(-1)^{1+1} \cdot 1 \begin{vmatrix} -6 & 4 \\ 12 & -4 \end{vmatrix} + (-1)^{3+1} \cdot 4 \begin{vmatrix} 3 & -1 \\ -6 & 4 \end{vmatrix} \right]$$

$$= 4 [(-12-12) - 3(-4)] - 6 [(24-48) + 4(12-6)]$$

$$= 4(-24+12) - 6(-24+24)$$

$$= -48.$$

so $|F| = -48$

$D \xrightarrow{-2R_2} F$ thus $|F| = -2|D| \Rightarrow |D| = -\frac{1}{2}|F| = 24$

$C \xrightarrow{R_3 \leftrightarrow R_2} D$ thus $|D| = -|C| \Rightarrow |C| = -24$

$B \xrightarrow{-6R_1 + R_4 \to R_4} C$ thus $|C| = |B| \Rightarrow |B| = -24$

$A \xrightarrow{3R_2} B$ thus $|B| = 3|A| \Rightarrow |A| = \frac{1}{3}|B| = -8$

5/5

* Question 6 :

i) $L = \{(a, b^3, 0) \mid a, b \in \mathbb{R}\}$

NO

$\in \mathbb{R}$

. axiom 1: let $Q_1, Q_2 \in L$ then $Q_1 = \alpha_1 a + \alpha_2 b^3 + 0$ for some const α_1, α_2

$Q_2 = \beta_1 a + \beta_2 b^3 + 0$ for some const. $\beta_1 \beta_2$

thus $Q_1 + Q_2 = (\alpha_1 + \beta_1) a + (\alpha_2 + \beta_2) b^3 + 0$

$\in \mathbb{R}$

not a point

so $Q_1 + Q_2 \in L$

. axiom 2: let $Q \in L$ and α be a constant

then $Q = \alpha_1 a + \alpha_2 b^3 + 0$

and $\alpha Q = (\alpha \alpha_1) a + (\alpha \alpha_2) b^3 + 0$

so $\alpha Q \in L$

X

0/3

. axiom 3: take $a = b = 0$ thus $(0,0,0) \in L$

So L is a subspace of $\mathbb{R}^3$.

(ii) $L = \{(a, 0, b^2) \mid a, b \in \mathbb{R}\}$

Take $Q = (0, 0, 4) \in L$ where $a = 0$ and $b = 2$

and consider $\alpha = -1$

3/3

thus $\alpha Q = (0, 0, -4) \notin L$

thus L is not a subspace of $\mathbb{R}^3$ since axiom 2 fails.

* Question 6:

iii) $L = \{ (b, b^3, 0) \mid b \in \mathbb{R} \}$

Take $Q = (1, 1, 0) \in L$ such that $b = 1$

and consider $\alpha = 2$

then $\alpha Q = (2, 2, 0) \notin L$

thus L is not a subspace of $\mathbb{R}^3$

M/M

iv) let us find the solution set of $(3I_4 - A)X = 0$

$$\begin{pmatrix} 3 & 0 & 0 & 0 \\ -1 & 3 & 0 & 0 \\ 0 & -1 & 3 & -4 \\ 0 & 0 & -1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 3 & 0 & 0 & 0 & 0 \\ -1 & 3 & 0 & 0 & 0 \\ 0 & -1 & 3 & -4 & 0 \\ 0 & 0 & -1 & 3 & 0 \end{array}\right) \xrightarrow[\cancel{\frac{1}{3}R_2}]{\frac{1}{3}R_1} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ -1 & 3 & 0 & 0 & 0 \\ 0 & -1 & 3 & -4 & 0 \\ 0 & 0 & -1 & 3 & 0 \end{array}\right)$$

$\downarrow R_1 + R_2 \to R_2$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 3 & -4 & 0 \\ 0 & 0 & -1 & 3 & 0 \end{array}\right) \xleftarrow{\frac{1}{3}R_2} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & -1 & 3 & -4 & 0 \\ 0 & 0 & -1 & 3 & 0 \end{array}\right)$$

$\downarrow R_2 + R_3 \to R_3$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & -4 & 0 \\ 0 & 0 & -1 & 3 & 0 \end{array}\right) \xrightarrow{\frac{1}{3}R_3} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -4/3 & 0 \\ 0 & 0 & -1 & 3 & 0 \end{array}\right)$$

$\xrightarrow{R_3+R_4\to R_4}$ $\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -4/3 & 0 \\ 0 & 0 & 0 & 5/3 & 0 \end{array}\right)$

7/7

$\xrightarrow{\frac{3}{5}R_4}$ $\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -4/3 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$ $\xrightarrow{\frac{4}{3}R_4+R_3\to R_3}$ $\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$

thus Solution set of $(3I_4 - A)X = 0$ is $\{(0,0,0,0)\}$

therefore 3 is not an eigenvalue of A.

v) Let A be a 4×4 matrix where $A_2 = A_4$ (columns)

solve the system $A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = A_2$

$\Rightarrow A_2 = x_1A_1 + x_2A_2 + x_3A_3 + x_4A_4$

$A_2 = x_1A_1 + (x_2+x_4)A_2 + x_3A_3$

thus $\begin{cases} x_1 = 0 \\ x_2 + x_4 = 1 \\ x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 0 \\ x_2 = 1 - x_4 \\ x_3 = 0 \end{cases} \xrightarrow{\text{take } x_4 = a} \begin{cases} x_1 = 0 \\ x_2 = 1-a \\ x_3 = 0 \\ x_4 = a \end{cases}$

thus the solution set of $AX = A_2 = \{(0, 1-a, 0, a) \mid a \in \mathbb{R}\}$

so it has infinitely many solutions

and $\underset{(a=1)}{(0,0,0,1)}$, $\underset{(a=2)}{(0,-1,0,2)}$, $\underset{(a=3)}{(0,-2,0,3)}$ are 3 distinct points that belong to the solution set of the system.

2.4 HW II

Assignment II MTH 512 , Fall 2019

Ayman Badawi

QUESTION 1. Let $F : R^4 \to R^3$, be a linear transformation. $B = \{(1,0,2,0),(0,1,1,0),(0,0,1,1),(-1,0,0,1)\}$ and $B' = \{(1,1,0),(-1,1,0),(-1,-1,1)\}$ be basis for R^4 and R^3, respectively. Given $F(1,0,2,0) = (1,-1,-1)$, $F(0,1,1,0) = (-1,0,1)$, $F(0,0,1,1) = (-2,0,2)$ and $F(-1,0,0,1) = (0,-1,0)$.

(i) Find the matrix presentation of F with respect to B and B', $M_{B,B'}$. (i.e., $M_{B,B'}$ = "something", I want to see that "something", however to calculate that "something" use software calculator as on I-learn)

(ii) USE (i) and find $[T(2,5,8,2)]_{B,B'}$

Note (again) write down clearly the steps, however use software calculator to do the actual calculation

(iii) Use (ii) and find $T(2,5,8,2)$.

(iv) Use (i) and find the standard matrix presentation. (I will not say it again, I want to see how you find M, actual calculations by software calculator)

QUESTION 2. Let $F : R^4 \to R^3$ such that $T(a_1, a_2, a_3, a_4) = (2a_1 + a_4, -a_3, 4a_1 + 2a_3 + a_4)$

(i) Write range(F) as span of some independent points.

(ii) Write range(F) as span of orthogonal points

(iii) Does the point $(2, 5, 9)$ belong to Range(F)? Explain?

(iv) Write Z(F) as span of some independent points

(v) Find the Standard matrix presentation of F.

(vi) Use (V) and find $T(-2,3,6,1)$

QUESTION 3. Let $F : R^3 \to R^4$ such that $T(2,0,0) = (1,1,1,1)$, $T(2,2,0) = (-2,-2,-2,-2)$, and $T(-1,-2,1) \in Z(F)$.

(i) Find the standard matrix presentation of F

(ii) write range of F as span of some independent points.

(iii) Write Z(F) as span of some independent points.

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2.5 Solution to HW II

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Assignment II MTH 512 , Fall 2019

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QUESTION 1. Let $F : R^4 \to R^3$, be a linear transformation. $B = \{(1,0,2,0),(0,1,1,0),(0,0,1,1),(-1,0,0,1)\}$ and $B' = \{(1,1,0),(-1,1,0),(-1,-1,1)\}$ be basis for R^4 and R^3, respectively. Given $F(1,0,2,0) = (1,-1,-1)$, $F(0,1,1,0) = (-1,0,1)$, $F(0,0,1,1) = (-2,0,2)$ and $F(-1,0,0,1) = (0,-1,0)$.

(i) Find the matrix presentation of F with respect to B and B', $M_{B,B'}$. (i.e., $M_{B,B'}$ = "something", I want to see that "something", however to calculate that "something" use software calculator as on I-learn)

$$\underset{3\times 4}{M_{BB'}} = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -1 & -2 & 0 \\ -1 & 0 & 0 & -1 \\ -1 & 1 & 2 & 0 \end{bmatrix}$$

4

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{-1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -2 & 0 \\ -1 & 0 & 0 & -1 \\ -1 & 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} -1 & \frac{1}{2} & 1 & -\frac{1}{2} \\ -1 & \frac{1}{2} & 1 & -\frac{1}{2} \\ -1 & 1 & 2 & 0 \end{bmatrix}$$

(ii) USE (i) and find $[T(2,5,8,2)]_{B,B'}$

Note (again) write down clearly the steps, however use software calculator to do the actual calculation

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$$B = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \longrightarrow B^{-1} = \begin{bmatrix} -1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 0 \\ 2 & 1 & -1 & 2 \\ -2 & -1 & 1 & -1 \end{bmatrix}$$

$$[2,5,8,2]_B = B^{-1}\begin{bmatrix} 2 \\ 5 \\ 8 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \\ 5 \\ -3 \end{bmatrix}, \quad [T(-1,5,5,-3)]_{B'} = M_{B,B'}\begin{bmatrix} -1 \\ 5 \\ 5 \\ -3 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \\ 16 \end{bmatrix}$$

(iii) Use (ii) and find $T(2,5,8,2)$.

4

$$T(2,5,8,2) = B'\begin{bmatrix} 10 \\ 10 \\ 16 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 10 \\ 10 \\ 16 \end{bmatrix} = \begin{bmatrix} -16 \\ 4 \\ 16 \end{bmatrix}$$

(iv) Use (i) and find the standard matrix presentation. (I will not say it again, I want to see how you find M, actual calculations by software calculator)

4

$$M = B' M_{B,B'} B^{-1} = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & \frac{1}{2} & 1 & -\frac{1}{2} \\ -1 & \frac{1}{2} & 1 & -\frac{1}{2} \\ -1 & 1 & 2 & 0 \end{bmatrix}\begin{bmatrix} -1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 0 \\ 2 & 1 & -1 & 2 \\ -2 & -1 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & -4 & 3 & -5 \\ 3 & 2 & -2 & 2 \\ 5 & 4 & -3 & 5 \end{bmatrix}$$

QUESTION 2. Let $F : R^4 \to R^3$ such that $T(a_1, a_2, a_3, a_4) = (2a_1 + a_4, -a_3, 4a_1 + 2a_3 + a_4)$

(i) Write range(F) as span of some independent points.

$$T(a_1,a_2,a_3,a_4) = \{a_1(2,0,4) + a_2(0,0,0) + a_3(0,-1,2) + a_4(1,0,1) \mid a_1,a_2,a_3,a_4 \in \mathbb{R}\}.$$
$$= span\{(2,0,4), (0,0,0), (0,-1,2), (1,0,1)\}$$
$$= span\{\underbrace{(2,0,4)}_{Q_1}, \underbrace{(0,-1,2)}_{Q_2}, \underbrace{(1,0,1)}_{Q_3}\}.$$

4 ✓

(ii) Write range(F) as span of orthogonal points

For this particular question
$= span\{e_1, e_2, e_3\}$

$$W_1 = Q_1 = (2,0,4)$$

4

$$W_2 = Q_2 - \frac{Q_2 \cdot W_1}{\|W_1\|^2} W_1 = (0,-1,2) - \frac{(0,-1,2)\cdot(2,0,4)}{\|(2,0,4)\|^2}(2,0,4) = \left(\frac{-4}{5}, -1, \frac{2}{5}\right).$$

$$W_3 = Q_3 - \frac{Q_3 \cdot W_2}{\|W_2\|^2} W_2 - \frac{Q_3 \cdot W_1}{\|W_1\|} W_1$$

$$= (1,0,1) - \frac{(1,0,1)\cdot(-\frac{4}{5},-1,\frac{2}{5})}{\|(-\frac{4}{5},-1,\frac{2}{5})\|^2}\left(\frac{-4}{5},-1,\frac{2}{5}\right) - \frac{(1,0,1)\cdot(2,0,4)}{\|(2,0,4)\|^2}(2,0,4) = \left(\frac{2}{9}, \frac{-2}{9}, \frac{-1}{9}\right)$$

(iii) Does the point (2, 5, 9) belong to Range(F)? Explain?

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$$\begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 4 & 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix} \to \left[\begin{array}{cccc|c} 2 & 0 & 0 & 1 & 2 \\ 0 & 0 & -1 & 0 & 5 \\ 4 & 0 & 2 & 1 & 9 \end{array}\right] \to \begin{cases} a_1 = \frac{17}{2} \\ a_2 = \text{free variable} \\ a_3 = -5 \\ a_4 = -15 \end{cases}$$

∴ YES. because there is at least one point $\in \mathbb{R}^4$ such that F(at this point) = (2,5,9).

Sol. set $= \{(\frac{17}{2}, a_2, -5, -15)\}$

(iv) Write Z(F) as span of some independent points

4

$$\begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 4 & 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \to \left[\begin{array}{cccc|c} 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 4 & 0 & 2 & 1 & 0 \end{array}\right] \to \begin{cases} a_1 = 0 \\ a_2 = \text{free variable} \\ a_3 = 0 \\ a_4 = 0 \end{cases}$$

$$\text{Sol. set} = \{(0, a_2, 0, 0) \mid a_2 \in \mathbb{R}\}$$
$$= \{a_2(0,1,0,0) \mid a_2 \in \mathbb{R}\}$$
$$= span\{(0,1,0,0)\}.$$

(v) Find the Standard matrix presentation of F.

4

$$M = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 4 & 0 & 2 & 1 \end{bmatrix}$$

(vi) Use (V) and find $T(-2, 3, 6, 1)$

4

$$\begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 4 & 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \\ 6 \\ 1 \end{bmatrix} = -2 \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + 6 \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ -6 \\ 5 \end{bmatrix}$$

QUESTION 3. Let $F : R^3 \to R^4$ such that $T(2,0,0) = (1,1,1,1)$, $T(2,2,0) = (-2,-2,-2,-2)$, and $T(-1,-2,1) \in Z(F)$. • T is a Linear Transformation .

(i) Find the standard matrix presentation of F

$* \; T(e_1) = T(1,0,0) = T\left(\frac{1}{2}(2,0,0)\right) = \frac{1}{2}T(2,0,0) = \frac{1}{2}(1,1,1,1) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right).$

$* \; T(e_2) = T(0,1,0) = T\left(-\frac{1}{2}(2,0,0) + \frac{1}{2}(2,2,0)\right) = -\frac{1}{2}T(2,0,0) + \frac{1}{2}T(2,2,0)$

$= -\frac{1}{2}(1,1,1,1) + \frac{1}{2}(-2,-2,-2,-2) = \left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right) + (-1,-1,-1,-1) = \left(\frac{-3}{2}, \frac{-3}{2}, \frac{-3}{2}, \frac{-3}{2}\right)$

$* \; T(e_3) = T(0,0,1) = T\left(-\frac{1}{2}(2,0,0) + (2,2,0) + (-1,-2,1)\right) = -\frac{1}{2}T(2,0,0) + T(2,2,0) + T(-1,-2,1)$

$= -\frac{1}{2}(1,1,1,1) + (-2,-2,-2,-2) + (0,0,0) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) + (-2,-2,-2,-2) = \left(\frac{-5}{2}, \frac{-5}{2}, \frac{-5}{2}, \frac{-5}{2}\right)$

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$$\therefore M = \begin{array}{c} \begin{array}{ccc} T(e_1) & T(e_2) & T(e_3) \end{array} \\ \begin{bmatrix} \frac{1}{2} & -\frac{3}{2} & -\frac{5}{2} \\ \frac{1}{2} & -\frac{3}{2} & -\frac{5}{2} \\ \frac{1}{2} & \frac{-3}{2} & \frac{-5}{2} \\ \frac{1}{2} & -\frac{3}{2} & \frac{-5}{2} \end{bmatrix} \end{array}$$

(ii) write range of F as span of some independent points.

4

$$\text{Range}(F) = \text{span}\left\{\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)\right\}$$

(iii) Write Z(F) as span of some independent points.

4

$$\begin{bmatrix} \frac{1}{2} & -\frac{3}{2} & -\frac{5}{2} \\ \frac{1}{2} & -\frac{3}{2} & -\frac{5}{2} \\ \frac{1}{2} & \frac{-3}{2} & \frac{-5}{2} \\ \frac{1}{2} & \frac{-3}{2} & \frac{-5}{2} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \to \left[\begin{array}{ccc|c} \frac{1}{2} & -\frac{3}{2} & -\frac{5}{2} & 0 \\ \frac{1}{2} & -\frac{3}{2} & -\frac{5}{2} & 0 \\ \frac{1}{2} & \frac{-3}{2} & \frac{-5}{2} & 0 \\ \frac{1}{2} & \frac{-3}{2} & \frac{-5}{2} & 0 \end{array}\right] \to \begin{array}{l} a_1 = 3a_2 + 5a_3 \\ a_2 : \text{free variable} \\ a_3 : \text{free variable.} \end{array}$$

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$$\begin{aligned} \text{Sol. set} &= \{(3a_2 + 5a_3, a_2, a_3) \mid a_2, a_3 \in \mathbb{R}\} \\ &= \{a_2(3,1,0) + a_3(5,0,1) \mid a_2, a_3 \in \mathbb{R}\} \\ &= \text{span}\{(3,1,0), (5,0,1)\} \end{aligned}$$

2.6 HW III

Assignment III, MTH 512 , Fall 2019

Ayman Badawi

QUESTION 1. Let $T : V \to W$ be a linear transformation between two vector spaces over R, say V and W.

(i) Prove that T is one-to-one if and only if $Z(T) = \{0_V\}$.

(ii) Assume that $T(v_0) = w_0$ for some $v_0 \in V$ and for some $w_0 \in W$. Prove that $T^{-1}(w_0) = \{v_0 + d \mid d \in Z(T)\}$.

(iii) Fix an integer $n \geq 1$, let $C^n[R]$ be the vector space of all continuous nth-derivative functions over R. (We know that $C^n[R]$ is a vector space, do not show that). Define $T : C^n[R] \to C^n[R]$ such that $T(y(x)) = a_n y^{(n)} + a_{n-1}y^{(n-1)} + ... + a_1 y' + a_0 y$, where $a_0, a_1, ..., a_n$ are some fixed real numbers. Show that T is a linear transformation (briefly). Let $d(x) \in C^n[R]$ such that $T(d(x)) = f(x)$. Show that $T^{-1}(f(x)) = \{d(x) + m \mid m \in Z(T)\}$. [Hint: just use (ii), **BIG THING: Now we all understand why when solving Linear Diff. Equation, then the solution is $y_h + y_p$, where y_h is the homogeneous part and y_p is the particular part**].

QUESTION 2. (a) Let $D = \{\begin{bmatrix} a+2b & 3a+c \\ 5a+4b+c & -2a-4b \end{bmatrix} \mid a, b, c \in R\}$. Convince me that D is a subspace of $R^{2\times 2}$. (I guess, it is enough to rewrite D as span). Then find IN(D) (dim(D)).

(b) Convince me that $D = \{(a+3b)x^3 + (-2a+b)x^2 + (-a+4b)x + (2a-b) \mid a, b \in R\}$ is a subspace of P_4. Find $IN(D)$.

QUESTION 3. Let $T : P_4 \to P_4$ such that $T(a_3x^3 + a_2x^2 + a_1x + a_0) = (a_2 - a_1 + a_0)x^2 + (2a_2 + a_0)x + (-a_2 + a_1 + 2a_0)$

(i) Find the fake standard matrix presentation of T.

(ii) Find Z(T).

(iii) Find Range(T).

(iv) Does $x^2 + 3x - 7$ belong to the RANGE(T)?Explain.

QUESTION 4. Let $T : P_3 \to R$ such that $T(x^2) = 1$, $T(2x) = 4$, $T(x+1) = -4$.
(a) Find the fake standard matrix presentation of T.
(b) Find Z(T).
(c) Let $H = \{a \in P_3 \mid T(a) = \pi\}$. Find the set H.

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2.7 Solution to HW III

Q1 $\frac{10}{12}$

(i) $T: V \to W$

$\leftarrow$ Suppose $Z(T) \in \{0_v\}$

$T(x) = T(y)$ for some $x, y \in V$

$\Rightarrow T(x) - T(y) = 0_W$

$\Rightarrow T(x-y) = 0_W \Rightarrow x-y \in Z(T)$

$\therefore x - y = 0_v \Rightarrow x = y$

$\rightarrow$ Suppose $T: V \to W$ is 1-1

let $x \in Z(T) \Rightarrow T(x) = 0_W = T(0_v)$

$\Rightarrow x = 0_v$ since T is 1-1

$\therefore Z(T) = \{0_v\}$

4/4

(ii) Let $M = \{v_0 + d \mid d \in Z(T)\}$

Show that $M \subseteq T^{-1}(w_0)$

Suppose $x \in M \Rightarrow x = v_0 + d$ where $d \in Z(T)$.

Then, $T(x) = T(v_0 + d)$

$= T(v_0) + T(d)$

$= w_0 + 0$

$= w_0$

$\Rightarrow T(x) = w_0 = T(v_0)$ & $x \in T^{-1}(w_0)$

Now we show that $T^{-1}(w_0) \subseteq M$

Suppose $x \in T^{-1}(w_0)$, then

$T(x - v_0) = T(x) - T(v_0)$ $\quad x, v_0 \in T^{-1}(w_0)$

$= w_0 - w_0$

$= 0$

$\Rightarrow x - v_0 \in Z(T) \Rightarrow \exists\, d \in Z(T)$ s.t. $d = x - v_0$

(i.e. $x = v_0 + d$)

$\therefore x \in M \Rightarrow T^{-1}(w_0) \subseteq M$ and $T^{-1}(w_0) = M$

4/4

Good

What is that?

~~[illegible]~~

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Question (1).

$T: V \to W$

(i) $(\rightarrow)$ Assume that T is one-to-one.

Show that $Z(T) = \{0_V\}$.

define $Z(T) = \{ V_0 \in V \mid T(V_0) = 0_W \}$, and we already Know that $T(0_V) = 0_W$;

Then $T(V_0) = T(0_V)$.

And since T is one-to-one;

Then $V_0 = 0_V$.

Therefore, $Z(T) = \{0_V\}$.

$(\leftarrow)$ Assume that $Z(T) = \{0_V\}$.

Show that T is one-to-one.

Let $V_1, V_2 \in V$ such that $T(V_1) = T(V_2)$.

Then we have:

$$T(V_1) - T(V_2) = 0_W$$

since T is a L.T $\rightarrow$ $T(V_1) + T(-V_2) = 0_W$

since T is a L.T $\rightarrow$ $T(V_1 - V_2) = 0_W$

Then, $V_1 - V_2 \in Z(T)$.

And since $Z(T) = \{0_V\}$, then $V_1 - V_2 = 0_V$.

Then, $V_1 = V_2$.

$\therefore$ Therefore, T is one-to-one.

(ii) Take $d \in V$ such that $T(d) = 0_W$.

That's $d \in Z(T)$.

Then $T(V_0 + d) = T(V_0) + T(d)$ ← since T is a L.T

$= W_0 + 0_W = W_0$.

$\therefore T(V_0+d) = W_0$

$\therefore T^{-1}(W_0) = V_0 + d$ one direction

Therefore, $T^{-1}(W_0) = \{ V_0 + d \mid d \in Z(T) \}$.

N/A

show $T^{-1}(W_0) \subseteq \{V_0+d \mid d \in Z(T)\}$

(iii) $T(y(x)) = a_{(n)} y^{(n)} + a_{(n-1)} y^{(n-1)} + \cdots + a_1 y' + a_0 y$.

Take $y_1, y_2 \in C^n[R]$ and $\alpha_1, \alpha_2 \in \mathbb{R}$.

To show that T is a linear transformation, it's enough to show $T(\alpha_1 y_1 + \alpha_2 y_2) = \alpha_1 T(y_1) + \alpha_2 T(y_2)$.

$$T(\alpha_1 y_1 + \alpha_2 y_2) = a_n [\alpha_1 y_1 + \alpha_2 y_2]^{(n)} + a_{n-1} [\alpha_1 y_1 + \alpha_2 y_2]^{(n-1)} + \cdots + a_0 [\alpha_1 y_1 + \alpha_2 y_2].$$

$$= \left(a_n \alpha_1 y_1^{(n)} + a_n \alpha_2 y_2^{(n)}\right) + \left(a_{n-1} \alpha_1 y_1^{(n-1)} + a_{n-1} \alpha_2 y_2^{(n-1)}\right) + \cdots + \left(a_0 \alpha_1 y_1 + a_0 \alpha_2 y_2\right).$$

$$= \left(a_n \alpha_1 y_1^{(n)} + a_{n-1} \alpha_1 y_1^{(n-1)} + \cdots + a_0 \alpha_1 y_1\right) + \left(a_n \alpha_2 y_2^{(n)} + a_{n-1} \alpha_2 y_2^{(n-1)} + \cdots + a_0 \alpha_2 y_2\right).$$

$$= \alpha_1 \left(a_n y_1^{(n)} + a_{n-1} y_1^{(n-1)} + \cdots + a_0 y_1\right) + \alpha_2 \left(a_n y_2^{(n)} + a_{n-1} y_2^{(n-1)} + \cdots + a_0 y_2\right).$$

$$= \alpha_1 T(y_1) + \alpha_2 T(y_2).$$

$\therefore$ Hence, T is a linear transformation.

Show that $T^{-1}(f(x)) = \{ d(x) + m \mid m \in Z(T) \}$.

Take $m \in C^n[R]$ such that $T(m) = 0_{C^n[R]}$.

That's $m \in Z(T)$

Then $T(d(x)+m) = T(d(x)) + T(m)$ since T is a L.T

$= f(x) + 0_{C^n[R]}$ since $T(d(x)) = f(x)$ (given).

$= f(x)$.

$\therefore T(d(x)+m) = f(x)$

$\therefore T^{-1}(f(x)) = d(x) + m$.

Therefore, $T^{-1}(f(x)) = \{ d(x) + m \mid m \in Z(T) \}$.

$1/2$

Question (2).

(a) $D = \left\{ \begin{bmatrix} a+2b & 3a+c \\ 5a+4b+c & -2a-4b \end{bmatrix} \middle| \; a,b,c \in \mathbb{R} \right\}$

$\mathbb{R}^{2\times 2} \cong \mathbb{R}^4$
(as vector space)

Fake set correspond to D:

$D' = \{ (a+2b,\ 3a+c,\ 5a+4b+c,\ -2a-4b) \mid a,b,c \in \mathbb{R} \}$

$= \{ a(1,3,5,-2) + b(2,0,4,-4) + c(0,1,1,0) \mid a,b,c \in \mathbb{R} \}$

$= \text{span}\{ \overset{dep}{(1,3,5,-2)}, (2,0,4,-4), (0,1,1,0) \}$

$= \text{span}\{ (2,0,4,-4), (0,1,1,0) \}$

$IN(D') = 2$

$D = \text{span}\left\{ \begin{bmatrix} 2 & 0 \\ 4 & -4 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$

$IN(D) = 2$.

∴ Therefore, D is a subspace of $\mathbb{R}^{2\times 2}$.

$$\begin{bmatrix} 1 & 3 & 5 & -2 \\ 2 & 0 & 4 & -4 \\ 0 & 1 & 1 & 0 \end{bmatrix} \xrightarrow{-2R_1+R_2} \begin{bmatrix} 1 & 3 & 5 & -2 \\ 0 & -6 & -6 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \xrightarrow{\frac{1}{6}R_2+R_3} \begin{bmatrix} 1 & 3 & 5 & -2 \\ 0 & -6 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

∴ 2 leaders

∴ IN = 2

(b)

$$D = \{ (a+3b)x^3 + (-2a+b)x^2 + (-a+4b)x + (2a-b) \mid a,b \in \mathbb{R} \}$$

$P_4 \cong \mathbb{R}^4$
(as vector space)

Fake set correspond to D :

$$D' = \{ (a+3b, -2a+b, -a+4b, 2a-b) \mid a,b \in \mathbb{R} \}$$

$$= \{ a(1,-2,-1,2) + b(3,1,4,-1) \mid a,b \in \mathbb{R} \}$$

$$= \text{span} \{ (1,-2,-1,2), (3,1,4,-1) \}$$

$\text{IN}(D') = 2.$

$$D = \text{span} \{ (x^3 - 2x^2 - x + 2), (3x^3 + x^2 + 4x - 1) \}$$

$\text{IN}(D) = 2.$

$\therefore$ Therefore, D is a subspace of P_4.

Question (3).

$T : P_4 \rightarrow P_3$

$T(a_3x^3 + a_2x^2 + a_1x + a_0) = (a_2 - a_1 + a_0)x^2 + (2a_2 + a_0)x + (-a_2 + a_1 + 2a_0)$

(i) $T' : \mathbb{R}^4 \rightarrow \mathbb{R}^3$

$$T'(a_0, a_1, a_2, a_3) = (-a_2 + a_1 + 2a_0,\ 2a_2 + a_0,\ a_2 - a_1 + a_0).$$

$$= \{a_0(2,1,1) + a_1(1,0,-1) + a_2(-1,2,1) + a_3(0,0,0,0)\}$$

$$= \text{span}\{(2,1,1), (1,0,-1), (-1,2,1), (0,0,0)\}$$

$$= \text{span}\{(2,1,1), (1,0,-1), (-1,2,1)\}.$$

$$M' = \begin{bmatrix} 2 & 1 & -1 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & -1 & 1 & 0 \end{bmatrix}$$

(ii) $Z(T')$: $M' \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & -1 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\left[\begin{array}{cccc|c} 2 & 1 & -1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \end{array}\right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{cccc|c} 1 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \end{array}\right] \xrightarrow[-R_1+R_3]{-R_1+R_2}$$

$$\left[\begin{array}{cccc|c} 1 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{5}{2} & 0 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & 0 & 0 \end{array}\right] \xrightarrow{-2R_2} \left[\begin{array}{cccc|c} 1 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & -5 & 0 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & 0 & 0 \end{array}\right] \xrightarrow{\frac{3}{2}R_2 + R_3}$$

$$\left[\begin{array}{cccc|c} 1 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & -5 & 0 & 0 \\ 0 & 0 & -6 & 0 & 0 \end{array}\right] \rightarrow \begin{array}{l} -6a_2 = 0 \rightarrow a_2 = 0 \\ a_1 - 5a_2 = 0 \rightarrow a_1 = 0 \\ a_0 + \frac{1}{2}a_1 - \frac{1}{2}a_2 = 0 \rightarrow a_0 = 0 \\ a_3 \text{ is a free variable} \end{array}$$

$$\text{Sol. set} = \{(0,0,0,a_3) \mid a_3 \in \mathbb{R}\}$$
$$= \{a_3(0,0,0,1) \mid a_3 \in \mathbb{R}\}$$

$$Z(T') = \text{span}\{(0,0,0,1)\}$$

$\therefore$ Sol. set of T = $\text{span}\{x^3\}$.

$\therefore$ $Z(T) = \text{span}\{x^3\}$

(iii) $\text{Range}(T') = \overset{\text{span}}{\text{Col}}(M')$

$$= \text{span}\{(1,1,2), (-1,0,1), (1,2,-1)\}$$

$$\text{Range}(T) = \text{span}\{(x^2+x+2,\ -x^2+1,\ x^2+2x-1)\}$$

(iv) Does $(-7,3,1)$ belong to $\text{Range}(T')$?

$$M'\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -7 \\ 3 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & -1 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & -1 & 1 & 0 \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -7 \\ 3 \\ 1 \end{bmatrix}$$

$\underset{T'}{\text{Sol. set}} = \{(-2, -\frac{1}{2}, \frac{5}{2}, a_3) \mid a_3 \in \mathbb{R}\} \rightarrow \underset{T}{\text{Sol. set}} = \{(-2 - \frac{1}{2}x + \frac{5}{2}x^2 + a_3x^3) \mid a_3 \in \mathbb{R}\}$

Since there exist (a_0, a_1, a_2, a_3) such that $(-7,3,1) \in \text{Range}(T')$,

then there exist a polynomaial $a_0 + a_1x + a_2x^2 + a_3x^3$ such that $-7+3x+x^2 \in \text{Range}(T)$.

$\therefore$ Answer is YES.

Question (4).

$T : P_3 \to \mathbb{R}$

$$\begin{cases} T(x^2) = 1 \\ T(2x) = 4 \\ T(x+1) = -4 \end{cases}$$

(a) $T' : \mathbb{R}^3 \to \mathbb{R}$

$$\begin{cases} T'(1,0,0) = 1 \\ T'(0,2,0) = 4 \\ T'(0,1,1) = -4 \end{cases}$$

$T'(e_1) = T'(1,0,0) = 1$

$T'(e_2) = T'(0,1,0) = T'\left(\frac{1}{2}(0,2,0)\right) = \frac{1}{2}T'(0,2,0) = \frac{1}{2}(4) = 2$.

$T'(e_3) = T'(0,0,1) = T'\left(-\frac{1}{2}(0,2,0) + (0,1,1)\right) = -\frac{1}{2}T'(0,2,0) + T'(0,1,1)$

$= -\frac{1}{2}(4) + (-4) = -2-4 = -6$.

$$\therefore M' = \begin{array}{c} \begin{array}{ccc} T'(e_1) & T'(e_2) & T'(e_3) \end{array} \\ \begin{bmatrix} 1 & 2 & -6 \end{bmatrix} \end{array}$$

(b)

$Z(T') \to \begin{bmatrix} 1 & 2 & -6 \end{bmatrix} \begin{bmatrix} a_2 \\ a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$

$\to \left[\begin{array}{ccc|c} 1 & 2 & -6 & 0 \end{array}\right]$

$\to \quad a_2 + 2a_1 - 6a_0 = 0$

$\to \quad a_2 = -2a_1 + 6a_0$, a_1 & a_2 are free variables.

Sol. set $Z(T') = \{ (-2a_1 + 6a_0, a_1, a) \mid a_1, a_0 \in \mathbb{R} \}$

$= \{ a_1(-2,1,0) + a_0(6,0,1) \mid a_1, a_0 \in \mathbb{R} \}$

$= \text{span}\{ (-2,1,0), (6,0,1) \}$

$Z(T) = \text{span}\{ (-2x^2 + x), (6x^2 + 1) \}$

2

(c) $H = \{ a \in P_3 \mid T(a) = \pi \}$

$$T(a) = \pi \rightarrow T(a_2x^2 + a_1x + a_0) = \pi$$

$$\rightarrow T'(a_2, a_1, a_0) = \pi$$

$$\rightarrow M' \begin{bmatrix} a_2 \\ a_1 \\ a_0 \end{bmatrix} = \pi$$

$$\rightarrow \begin{bmatrix} 1 & 2 & -6 \end{bmatrix} \begin{bmatrix} a_2 \\ a_1 \\ a_0 \end{bmatrix} = \pi$$

4/4

$$\rightarrow a_2 + 2a_1 - 6a_0 = \pi$$

$\rightarrow a_2 = \pi - 2a_1 + 6a_0$, a_1 & a_2 are free variables.

$$\rightarrow \text{Sol. set } (T') = \{ (\pi - 2a_1 + 6a_0, a_1, a_0) \mid a_1, a_0 \in \mathbb{R} \}$$

$$\rightarrow \text{Sol. set } (T) = \{ ((\pi - 2a_1 + 6a_0)x^2 + a_1x + a_0) \mid a_1, a_0 \in \mathbb{R} \}$$

$$\therefore H = \{ (\pi - 2a_1 + 6a_0)x^2 + a_1x + a_0 \mid a_1, a_0 \in \mathbb{R} \}$$

note that $T(x^2) = 1$. Hence $T(\pi x^2) = \pi$.

By Question ~~[illegible]~~ 1(c)(i): $T^{-1}(\pi) = \{ \pi x^2 + m \mid m \in Z(T) \}$

2.8 HW IV

Name———————————————, ID ———————

Assignment, IV, MTH 512 , Fall 2019

Ayman Badawi

QUESTION 1. Form a basis for $Hom(P_2, R^{2\times 2})$.

QUESTION 2. Let V be a vector space such that $IN(V) = 8$. Given W, K are subspaces of V such that $IN(W) = 5$ and $IN(K) = 4$. Find all possibilities of $IN(W \cap K)$. (Note that IN() = dim())

QUESTION 3. Let $T : P^4 \to R^3$ be a linear transformation such that $T(f(x)) = (\int_0^1 f(x)\, dx, f'(0), 0)$

(0.5a) Find the fake standard matrix presentation of T.

(a) Find Range(T) [Hint: one way is to find the fake T']

(b) Find Z(T) [Hint: again, you may make use of T']

QUESTION 4. Let $T : R^4 \to R^4$ such that $T(a_1, a_2, a_3, a_4) = (2a_1 + a_3, 0, a_1, a_1)$ and $F : R^4 \to R^4$ such that $F(b_1, b_2, b_3, b_4) = (b_1 + b_2, -3b_1 - 3b_2, b_3, 4b_3)$. Then we know that $T + F : R^4 \to R^4$ is a linear transformation.

(0.5a) Find the standard matrix presentation of $T + F$

(a) Find $Range(T + F)$

(b) Find $Z(T + F)$

(1.5b) Find the standard matrix presentation of T^2

(c) Find $Range(T^2)$

QUESTION 5. (a) A matrix A, $n\times n$, is called an idempotent matrix if $A^2 = A$. Assume that A is a nontrivial idempotent matrix (note that the zero-matrix $n \times n$ and I_n are called trivial idempotents). Convince me that the homogeneous system $AX = 0$ has infinitely many solutions.

(0.3a) Let A be an idempotent matrix, $n \times n$. Convince me that $I - A$ is an idempotent matrix.

(b) A matrix A, $n \times n$, is called a nilpotent matrix if $A^m = 0 - Matrix$, for some positive integer m. Convince me that $A + I_n$ is an invertible matrix.

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2.9 Solution to HW IV

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1

$\frac{52}{52}$

Assignment IV

* Question 1: Form a basis for Hom $(P_2, \mathbb{R}^{2\times 2})$

. $T_1' : \mathbb{R}^2 \longrightarrow \mathbb{R}^4$ where:

. $T_1'(a_1, a_2) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \sim T_1(a_1 + a_2 x) = \begin{pmatrix} a_1 & 0 \\ 0 & 0 \end{pmatrix}$

. $T_2'(a_1, a_2) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \sim T_2(a_1 + a_2 x) = \begin{pmatrix} a_2 & 0 \\ 0 & 0 \end{pmatrix}$

. $T_3'(a_1, a_2) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ a_1 \\ 0 \\ 0 \end{bmatrix} \sim T_3(a_1 + a_2 x) = \begin{pmatrix} 0 & a_1 \\ 0 & 0 \end{pmatrix}$

. $T_4'(a_1, a_2) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ a_2 \\ 0 \\ 0 \end{bmatrix} \sim T_4(a_1 + a_2 x) = \begin{pmatrix} 0 & a_2 \\ 0 & 0 \end{pmatrix}$

. $T_5'(a_1, a_2) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ a_1 \\ 0 \end{bmatrix} \sim T_5(a_1 + a_2 x) = \begin{pmatrix} 0 & 0 \\ a_1 & 0 \end{pmatrix}$

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$$\cdot\ T_6'(a_1,a_2) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ a_2 \\ 0 \end{bmatrix} \sim T_6(a_1 + a_2 X) = \begin{pmatrix} 0 & 0 \\ a_2 & 0 \end{pmatrix}$$

$$\cdot\ T_7'(a_1,a_2) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ a_1 \end{bmatrix} \sim T_7(a_1 + a_2 X) = \begin{pmatrix} 0 & 0 \\ 0 & a_1 \end{pmatrix}$$

$$\cdot\ T_8'(a_1,a_2) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ a_2 \end{bmatrix} \sim T_8(a_1 + a_2 X) = \begin{pmatrix} 0 & 0 \\ 0 & a_2 \end{pmatrix}$$

So the basis of $Hom(P_2, \mathbb{R}^{2\times 2}) = span\{T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8\}$

✓

2

* Question 2:

Let V be a vector space s.t IN (V) = 8

Let W and K be subspaces of V s.t IN (W) = 5 and IN (K) = 4

- We proved in class that W + K is a subspace of V

then IN (W + K) $\leq$ IN (V)

$\Rightarrow$ IN (W) + IN (K) − IN ($W \cap K$) $\leq$ IN(V)

9 − IN ($W \cap K$) $\leq$ 8

IN ($W \cap K$) $\geq$ 1

and IN ($W \cap K$) can't be greater than ~~(or equal to)~~ IN (W) and IN (K)

thus IN ($W \cap K$) $\leq$ 4

so 1 $\leq$ IN ($W \cap K$) $\leq$ 4

therefore IN ($W \cap K$) = 1 or 2 or 3 or 4

* Question 3: Let $T: P^3 \to \mathbb{R}^3$ be a L.T

s.t $T(f(x)) = \left(\int_0^1 f(x)\,dx ,\ f'(0),\ 0 \right)$

So $T(a_1 + a_2 X + a_3 X^2 + a_4 X^3) = \left(a_1 + \frac{1}{2}a_2 + \frac{1}{3}a_3 + \frac{1}{4}a_4 ,\ a_2 ,\ 0\right)$

a) First find the fake L.T $T': \mathbb{R}^4 \to \mathbb{R}^3$ s.t:

$$T'(a_1, a_2, a_3, a_4) = \left(a_1 + \frac{1}{2}a_2 + \frac{1}{3}a_3 + \frac{1}{4}a_4 ,\ a_2,\ 0\right)$$

so the standard matrix presentation is:

$$M = \begin{array}{c} \begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \end{array} \\ \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

b) $\text{Range}(T') = \left\{ \left(a_1 + \frac{1}{2}a_2 + \frac{1}{3}a_3 + \frac{1}{4}a_4 ,\ a_2 ,\ 0\right) \mid a_1, a_2, a_3, a_4 \in \mathbb{R} \right\}$

$$= \left\{ a_1(1,0,0) + a_2\left(\tfrac{1}{2},1,0\right) + a_3\left(\tfrac{1}{3},0,0\right) + a_4\left(\tfrac{1}{4},0,0\right) \mid a_1, a_2, a_3, a_4 \in \mathbb{R} \right\}$$

$$= \text{span}\left\{ (1,0,0), \left(\tfrac{1}{2},1,0\right) \right\}$$

and $\text{Range}(T) = \text{span}\left\{ (1,0,0), \left(\tfrac{1}{2},1,0\right) \right\}$

c) To find $Z(T)$, first we will solve the system $M \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{cases} a_2 = 0 \\ a_1 = -\frac{1}{3}a_3 - \frac{1}{4}a_4 \end{cases}$$

so $Z(T') = \left\{ \left(-\frac{1}{3}a_3 - \frac{1}{4}a_4 ,\ 0,\ a_3,\ a_4\right) \mid a_3, a_4 \in \mathbb{R} \right\}$

$$= \text{span}\left\{ \left(-\tfrac{1}{3}, 0, 1, 0\right), \left(-\tfrac{1}{4}, 0, 0, 1\right) \right\}$$

thus $Z(T) = \text{span}\left\{ \left(-\frac{1}{3} + X^2\right), \left(-\frac{1}{4} + X^3\right) \right\}$

* Question 4: $T: \mathbb{R}^4 \to \mathbb{R}^4$ s.t $T(a_1, a_2, a_3, a_4) = (2a_1 + a_3, 0, a_1, a_1)$
and $F: \mathbb{R}^4 \to \mathbb{R}^4$ s.t $F(b_1, b_2, b_3, b_4) = (b_1 + b_2, -3b_1 - 3b_2, b_3, 4b_3)$

a) $T+F: \mathbb{R}^4 \to \mathbb{R}^4$ is a L.T s.t:

$$(T+F)(c_1, c_2, c_3, c_4) = T(c_1, c_2, c_3, c_4) + F(c_1, c_2, c_3, c_4)$$
$$= (2c_1 + c_3, 0, c_1, c_1) + (c_1 + c_2, -3c_1 - 3c_2, c_3, 4c_3)$$
$$= (3c_1 + c_2 + c_3, -3c_1 - 3c_2, c_1 + c_3, c_1 + 4c_3)$$

and $M_T = \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$, $M_F = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -3 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & 0 \end{pmatrix}$

thus $M_{T+F} = M_T + M_F = \begin{pmatrix} 3 & 1 & 1 & 0 \\ -3 & -3 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 4 & 0 \end{pmatrix}$

4/4

b) $\text{Range}(T+F) = \text{span}\{(3, -3, 1, 1), (1, -3, 0, 0), (1, 0, 1, 4)\}$

c) To find $Z(T+F)$ we must solve the system $M_{T+F} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

thus $\left(\begin{array}{cccc|c} 3 & 1 & 1 & 0 & 0 \\ -3 & -3 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 4 & 0 & 0 \end{array}\right) \xrightarrow[-R_3 + R_4 \to R_4]{R_1 + R_2 \to R_2} \left(\begin{array}{cccc|c} 3 & 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \end{array}\right)$

Thus $C_3 = C_2 = C_1 = 0$

so $Z(T+F) = \{ (0,0,0,c_4) \mid c_4 \in \mathbb{R} \}$

$= \{ c_4 (0,0,0,1) \mid c_4 \in \mathbb{R} \}$

$\frac{4}{4}$ $= \text{Span} \{ (0,0,0,1) \}$

d) we know that $M_{T^2} = (M_T)^2 = M_T \cdot M_T$

thus $M_{T^2} = \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

$\frac{4}{4}$ $= \begin{pmatrix} 5 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 \end{pmatrix}$

e) $\text{Range}(T^2) = \text{span} \{ (5,0,2,2), (2,0,1,1) \}$

$\frac{4}{4}$

* Question 5:

5

a) Let A be an idempotent matrix i.e $A^2 = A$.

Case 1) $|A| \neq 0$

$\Leftrightarrow A$ is invertible

$\Leftrightarrow \exists A^{-1}$ s.t $\quad A^2 = A \Rightarrow \underbrace{A^{-1}A}A = A^{-1}A$

$\Rightarrow A = I_n$.

OK I see it

why!

i.e., why I_n is the only invertible idempotent?

Case 2) suppose $A \neq I_n$ then $|A| = 0$

$\Rightarrow A$ is non-invertible

$\Rightarrow \exists X \in \mathbb{R}^n$ s.t $AX = 0$ and $X \neq 0$

$\Rightarrow$ the system has infinitely many solutions since every scalar multiplication of X is a solution to the system.

X/X

b) $(I_n - A)^2 = I_n^2 - 2I_nA + A^2$

$= I_n - 2A + A$

$= I_n - A$

X/X

therefore $(I_n - A)$ is idempotent matrix.

~~Proof (Trivial). Suppose A is invertible and~~ $\cancel{A^2 = A}$ ~~. Hence~~ $\cancel{A^{-1}A^2 = A^{-1}A}$.

~~Thus~~ $\cancel{A^{-1}AA = A^{-1}A = I_n}$.

~~Hence~~ $\cancel{A = I_n}$.

c) let A, $n \times n$, be a nilpotent matrix i.e $A^m = 0$

suppose that $|A + I_n| = 0$

then $(A + I_n) V = 0$ for some $V \neq \vec{0} \in \mathbb{R}^n$

$\Rightarrow \quad AV = (-1) V$

$\Rightarrow \quad A^m V = (-1)^m V$

$\Rightarrow \quad A^m = (-1)^m I_n$ which contradicts $A^m = 0$

therefore $|A + I_n| \neq 0$

which implies that $A + I_n$ is invertible.

$\frac{4}{4}$

2.10 HW V

Name————————————, ID ——————

Assignment, V, MTH 512 , Fall 2019

Ayman Badawi

QUESTION 1. Let $T : V \to V$ be a linear transformation. Given $B = \{b_1, b_2, b_3, b_4\}$ is a basis for V such that $T(b_1) = b_2, T(b_2) = b_3, T(b_3) = b_4, T(b_4) = -b_1 + 2b_3$.

(i) Find M_B.

(ii) Convince me that $T^{-1} : V \to V$ exists. Then find $T^{-1}(b_1), T^{-1}(b_2), T^{-1}(b_3), T^{-1}(b_4)$. [Note that T^{-1} exists iff T is one-to-one and ONTO iff $|M_B| \neq 0$]

(iii) Find all eigenvalues of T . For each eigenvalue a of T, find $E_a(T) = \{v \in V \mid T(v) = av\}$, and write it as span.

(iv) Find all eigenvalues of T^{-1} . For each eigenvalue w of T^{-1}, find $E_w(T^{-1})$ and write it as span.

(v) Find $C_T(\alpha)$ and $m_T(alpha)$.

(vi) Convince me that T is not diagnolizable.

(vii) Find $C_T^{-1}(\alpha)$ and $m_{T^{-1}}(\alpha)$.

(viii) Define $F : V \to V$ such that $F(v) = -T^4(v) + 2T^2(v)$ for every $v \in V$. Then F is a linear transformation (DO NOT SHOW THAT). With minimum calculation, convince me that $F(v) = v$ for every $v \in V$, i.e., F is the identity map on V.

(ix) Let $F : V \to V$ such that $F(v) = T + I$ for every $v \in V$. Then F is a linear transformation (DO NOT SHOW THAT). With minimum calculation, convince me that F^{-1} does not exist.

QUESTION 2. Let T be a linear transformation from V into V such that $IN(V) = 5$ (note that IN(V) = dim(V)). Convince me that there exists a real number α and a nonzero element $v \in V$ such that $T(v) = \alpha v$.

QUESTION 3. Give me an example of a matrix A, 3×3, such that $C_A(\alpha) = m_A(\alpha)$ and A is not diagnolizable.

QUESTION 4. Give me an example of a matrix A, 3×3, such that $C_A(\alpha) = m_A(\alpha)$ and A is diagnolizable.

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2.11 Solution to HW V

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Assignment V

* Question 1: let $T: V \to V$ be a L.T.

$B = \{b_1, b_2, b_3, b_4\}$ is a basis of V such that:

. $T(b_1) = b_2$, $T(b_2) = b_3$, $T(b_3) = b_4$, $T(b_4) = -b_1 + 2b_3$

i) M_B :

$$\begin{array}{cccc} T(b_1) & T(b_2) & T(b_3) & T(b_4) \end{array}$$

$$M_B = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

ii) First let us compute $|M_B|$:

$$|M_B| = \begin{vmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{vmatrix} = 1 \neq 0$$

thus T is invertible $\Rightarrow T^{-1}$ exists.

and, . $T^{-1}(b_2) = b_1$

. $T^{-1}(b_3) = b_2$

. $T^{-1}(b_4) = b_3$

. $b_4 = T^{-1}(-b_1 + 2b_3) \Rightarrow b_4 = -T^{-1}(b_1) + 2T^{-1}(b_3)$

$\Rightarrow T^{-1}(b_1) = 2b_2 - b_4$

iii) Find eigenvalues of T:

First we will get the eigenvalues of M_B.

$C_{M_B}(a) = a^4 - 2a^2 + 1$ (since M_B is a companion matrix)

$= (a^2 - 1)^2$

$= (a-1)^2 (a+1)^2$

thus eigenvalues of M_B are 1 and -1

Hence eigenvalues of T are 1 and -1

. Now let us find the eigenspaces corresponding to the eigenvalues.

* $E_1(M_B)$: $(I_4 - M_B)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & -2 & 0 \\ 0 & 0 & -1 & 1 & 0 \end{array}\right] \xrightarrow{R_1 + R_2 \to R_2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & -2 & 0 \\ 0 & 0 & -1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 + R_3 \to R_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 \end{array}\right] \xrightarrow{R_3 + R_4 \to R_4} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Thus $x_1 = -x_4$, $x_2 = -x_4$, $x_3 = x_4$

and $E_1(M_B) = \{(-x_4, -x_4, x_4, x_4) \mid x_4 \in \mathbb{R}\}$

$= \text{span}\{(-1, -1, 1, 1)\}$

Hence $E_1(T) = \text{span}\{(-b_1 - b_2 + b_3 + b_4)\}$

$\ast$ $E_{-1}(M_B)$: $(-I_4 - M_B)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$$\left[\begin{array}{cccc|c} -1 & 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & -2 & 0 \\ 0 & 0 & -1 & -1 & 0 \end{array}\right] \xrightarrow{-R_1+R_2 \to R_2} \left[\begin{array}{cccc|c} -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & -1 & -1 & -2 & 0 \\ 0 & 0 & -1 & -1 & 0 \end{array}\right]$$

$$\xrightarrow{-R_2+R_3 \to R_3} \left[\begin{array}{cccc|c} -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & -1 & 0 \end{array}\right] \xrightarrow{-R_3+R_4 \to R_4} \left[\begin{array}{cccc|c} -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

thus $x_1 = x_4$, $x_2 = -x_4$, $x_3 = -x_4$

So $E_{-1}(M_B) = \{(x_4, -x_4, -x_4, x_4) \mid x_4 \in \mathbb{R}\}$

$= \text{span}\{(1,-1,-1,1)\}$

hence $E_{-1}(T) = \text{span}\{(b_1 - b_2 - b_3 + b_4)\}$.

iv) It is obvious that M_B^{-1} is the matrix presentation of T^{-1} with respect to the basis B.

Recall that if λ is an eigenvalue of M_B then $\frac{1}{\lambda}$ is an eigenvalue of M_B^{-1} $(\lambda \neq 0)$

Moreover : $E_\lambda(M_B) = E_{1/\lambda}(M_B^{-1})$

thus eigenvalues of T^{-1} are 1 and -1

and $E_1(T^{-1}) = E_1(T) = \text{span}\{-b_1 - b_2 + b_3 + b_4\}$

$E_{-1}(T^{-1}) = E_{-1}(T) = \text{span}\{b_1 - b_2 - b_3 + b_4\}$

v). We know that $C_T(\alpha) = C_{M_B}(\alpha)$

thus $C_T(\alpha) = (\alpha - 1)^2 (\alpha + 1)^2$

- we can notice that M_B is a companion matrix

thus $m_{M_B}(\alpha) = C_{M_B}(\alpha) = (\alpha-1)^2(\alpha+1)^2$

and $m_T(\alpha) = m_{M_B}(\alpha) = (\alpha-1)^2(\alpha+1)^2$

vi) Since $m_T(\alpha) = (\alpha-1)^2(\alpha+1)^2 \neq (\alpha-1)(\alpha+1)$

we can conclude that T is not diagonalizable.

vii) $$M_B^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

thus $$C_{M_B^{-1}}(\alpha) = |\alpha I_4 - M_B^{-1}| = \begin{vmatrix} \alpha & -1 & 0 & 0 \\ -2 & \alpha & -1 & 0 \\ 0 & 0 & \alpha & -1 \\ 1 & 0 & 0 & \alpha \end{vmatrix}$$

$$= \alpha \begin{vmatrix} \alpha & -1 & 0 \\ 0 & \alpha & -1 \\ 0 & 0 & \alpha \end{vmatrix} + 1 \begin{vmatrix} -2 & -1 & 0 \\ 0 & \alpha & -1 \\ 1 & 0 & \alpha \end{vmatrix}$$

$$= \alpha[\alpha(\alpha^2)] + 1[-2(\alpha^2) + 1(1)]$$

$$= \alpha^4 - 2\alpha^2 + 1$$

thus $C_{T^{-1}}(\alpha) = C_T(\alpha) = \alpha^4 - 2\alpha^2 + 1 = (\alpha-1)^2(\alpha+1)^2$

- we know that T^{-1} is not diagonalizable since T is not diagonalizable thus $m_{T^{-1}}(\alpha) \neq (\alpha-1)(\alpha+1)$

$\longrightarrow$

but the matrix presentation of T^{-1} with respect to some basis is not a companion matrix.

thus $m_{T^{-1}}(\alpha) = C_{T^{-1}}(\alpha)$ or $(\alpha-1)^2(\alpha+1)$ or $(\alpha-1)^2$ or $(\alpha+1)$ or $(\alpha-1)$ or $(\alpha+1)^2$

viii) Let $F: V \longrightarrow V$ s.t $F(v) = -T^4(v) + 2T^2(v) \quad \forall v \in V$

we know that $C_T(\alpha)|_{\alpha=T} = 0$ - function

thus $T^4 - 2T^2 + I = 0$ where I is the identity map on V

$\Leftrightarrow -T^4 + 2T^2 - I = 0$

$\Rightarrow \underbrace{-T^4(v) + 2T^2(v)} - I(v) = 0$

$\Rightarrow F(v) - v = 0$

$\Rightarrow F(v) = v$ for every $v \in V$

(ix) Recall that -1 is an eigenvalue of T

then $T(v) = -v$

$\Rightarrow T = -I$ where I is the identity map on V

$\Rightarrow T + I = 0$

$\Rightarrow F = 0$ - function

and It is obvious that F is not invertible

thus F^{-1} is doesn't exist.

* Question 2:

Let $T: V \to V$ such that $IN(V) = 5$

. we know that the degree of the characteristic polynomial is equal to the dimesion of V.

thus $C_T(\alpha)$ has a degree 5.

So $C_T(\alpha)$ ~~has~~ is a polynomial of odd degree and we know that every polynomial of odd degree ~~has~~ must have at least one real root.

Thus T must have at least one real eigenvalue say α. and a corresponding eigenfunction $v \in V$, $v \neq 0$,

such that $T(v) = \alpha v$

* Question 3:

let $A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 0 & 3 \\ 0 & 1 & 0 \end{pmatrix}$

since A is a companion matrix

then $C_A(\alpha) = m_A(\alpha) = \alpha^3 - 3\alpha + 2$

$= (\alpha - 1)^2 (\alpha + 2)$

and since $m_A(\alpha) \neq (\alpha - 1)(\alpha + 2)$ thus A is not diagonalizable.

therefore A is 3×3 matrix, such that $C_A(\alpha) = m_A(\alpha)$ and A is not diagonalizable.

* Question 4:

$$\det A = \begin{pmatrix} 0 & 0 & 6 \\ 1 & 0 & -11 \\ 0 & 1 & 6 \end{pmatrix}$$

since A is a companion matrix we have:

$$C_A(\alpha) = m_A(\alpha) = \alpha^3 - 6\alpha^2 + 11\alpha - 6$$
$$= (\alpha - 1)(\alpha - 2)(\alpha - 3)$$

and A is diagonalizable. ~~since~~

thus A is a 3x3 matrix, such that $C_A(\alpha) = m_A(\alpha)$

and A is diagonalizable.

2.12 HW VI

Name——————————————, ID ————————

Assignment VI, MTH 512 , Fall 2019

Ayman Badawi

QUESTION 1. Let $T : R^3 \to R^3$ such that $T(a, b, c) = (a + b, 3c + 2a, 6c + 4a)$ Find a formula for T^*.

QUESTION 2. Let $M = span\{1, x^2\}$. Then M is a subspace of P_4. Define $< \ , \ >$ on P_4 such that $< f_1, f_2 >= \int_0^1 f_1 f_2 \, dx$. Find $M^\perp$. Note that $M^\perp$ is a subspace of P_4

QUESTION 3. Define $< \ \ >$ on R^2 such that $< (a_1, b_1), (a_2, b_2) >= a_1 a_2 + 0.5(a_1 b_2 + a_2 b_1) + \frac{1}{3} b_1 b_2$. Convince me that $< \ \ >$ is an inner product on R^2.[Hint: One way is to verify the 3 axioms...boring calculations or stare a little: Observe that P^2 is R^2 as vector spaces (a, b) is a + bx in P_2, also $< f_1, f_2 >= \int_0^1 f_1 f_2 \, dx$ is an inner product on P_2. Now translate this inner product to R^2. Done]

QUESTION 4. Let $a_1, a_2, ..., a_5, b_1, b_2, ..., b_5$ be some real numbers. Convince me that $(a_1 b_1 + a_2 b_2 + ..., a_5 b_5)^2 \leq (a_1^2 + a_2^2 + ... + a_5^2)(b_1^2 + b_2^2 + ... + b_5^2)$

QUESTION 5. Let V be an inner product space. Convince me that $||v + w|| \leq ||v|| + ||w||$ for every $v, w \in V$

QUESTION 6. Let $D = Span\{1, x^3, x^4\}$. Find an orthonormal basis of D, where $< f_1, f_2 >= \int_0^1 f_1 f_2 \, dx$. [Note you will use the same idea as we did in dot product earlier, but here use $< \ , \ >$, so $w_1 = 1$, $w_2 = v_2 - \frac{< v_2, w_1 >}{||w_1||^2} w_1$ (v_2 here is x^3) and so on... same algorithm as in the case of dot product. Then make them Orthonormal.

QUESTION 7. Given $\begin{bmatrix} a & -3 \\ b & c \end{bmatrix}$ is positive definite. Find all possible values of a, b, c.

QUESTION 8. Given $1 - 2x, v_2, v_3, v_4$ is an orthogonal basis of P_4, where $< f_1, f_2 >= \int_0^1 f_1 f_2 \, dx$. Then $4x^3 = c_1(1 - 2x) + c_2 v_2 + c_3 v_3 + c_4 v_4$. Find the value of c_1.

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2.13 Solution to HW VI

MTH 512 - HW6

Fatimah Abdullah

-900085282-

Question(1).

Find M_T:

$T(e_1) = T(1,0,0) = (1,2,4)$

$T(e_2) = T(0,1,0) = (1,0,0)$

$T(e_3) = T(0,0,1) = (0,3,6)$

$$\therefore M_T = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 3 \\ 4 & 0 & 6 \end{bmatrix}$$

Define $T^*: \mathbb{R}^3 \to \mathbb{R}^3$.

$$M_{T^*} = (M_T)^T = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 0 & 0 \\ 0 & 3 & 6 \end{bmatrix}$$

$$T^*(v) = M_{T^*} v$$

Hence, $$T^*(a,b,c) = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 0 & 0 \\ 0 & 3 & 6 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$= \begin{bmatrix} a + 2b + 4c \\ a \\ 3b + 6c \end{bmatrix}$$

$$\therefore T^*(a,b,c) = (a+2b+4c,\ a,\ 3b+6c).$$

Question(2).

Let $v \in M^{\perp}$. Then $\langle 1, v \rangle = 0$ and $\langle x^2, v \rangle = 0$

Since $v \in M^{\perp}$ and $M^{\perp}$ is a subspace of P_4,

then $v \in P_4$.

Hence, $v = a_0 + a_1 x + a_2 x^2 + a_3 x^3$.

Then we have:

$$\langle 1, v \rangle = \langle 1, a_0 + a_1 x + a_2 x^2 + a_3 x^3 \rangle$$

$$= \int_0^1 (a_0 + a_1 x + a_2 x^2 + a_3 x^3)\, dx$$

$$= \left[a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3} + a_3 \frac{x^4}{4} \right]_0^1$$

$$= a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \frac{a_3}{4} = 0$$

$$\langle x^2, v \rangle = \langle x^2, a_0 + a_1 x + a_2 x^2 + a_3 x^3 \rangle$$

$$= \int_0^1 (x^2)(a_0 + a_1 x + a_2 x^2 + a_3 x^3)\, dx$$

$$= \int_0^1 (a_0 x^2 + a_1 x^3 + a_2 x^4 + a_3 x^5)\, dx$$

$$= \left[\frac{a_0 x^3}{3} + a_1 \frac{x^4}{4} + a_2 \frac{x^5}{5} + a_3 \frac{x^6}{6} \right]_0^1$$

$$= \frac{a_0}{3} + \frac{a_1}{4} + \frac{a_2}{5} + \frac{a_3}{6} = 0$$

Now, $M^{\perp}$ is the solution set of

$$\begin{cases} a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \frac{a_3}{4} = 0 \\ \frac{a_0}{3} + \frac{a_1}{4} + \frac{a_2}{5} + \frac{a_3}{6} = 0 \end{cases}$$

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \left[\begin{array}{cccc|c} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & 0 \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & 0 \end{array}\right]$$

$$\xrightarrow{REF} \left[\begin{array}{cccc|c} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & 0 \\ 0 & 1 & \frac{16}{15} & 1 & 0 \end{array}\right]$$

$$a_0 + \frac{1}{2}a_1 + \frac{1}{3}a_2 + \frac{1}{4}a_3 = 0 \rightarrow a_0 = -\frac{1}{2}a_1 - \frac{1}{3}a_2 - \frac{1}{4}a_3$$

$$a_1 + \frac{16}{15}a_2 + a_3 = 0 \rightarrow a_1 = -\frac{16}{15}a_2 - a_3$$

$$a_0 = -\frac{1}{2}\left(-\frac{16}{15}a_2 - a_3\right) - \frac{1}{3}a_2 - \frac{1}{4}a_3$$

a_2 is a free variable

$$a_0 = \frac{1}{5}a_2 + \frac{1}{4}a_3$$

a_3 is a free variable.

so, Sol. set $= \left\{ \left(\frac{1}{5}a_2 + \frac{1}{4}a_3, \frac{-16}{15}a_2 - a_3, a_2, a_3 \right) \mid a_2, a_3 \in \mathbb{R} \right\}$

$= \left\{ a_2\left(\frac{1}{5}, \frac{-16}{15}, 1, 0\right) + a_3\left(\frac{1}{4}, -1, 0, 1\right) \mid a_2, a_3 \in \mathbb{R} \right\}$

$= \operatorname{span}\left\{ \left(\frac{1}{5}, \frac{-16}{15}, 1, 0\right), \left(\frac{1}{4}, -1, 0, 1\right) \right\}$

$\therefore M^{\perp} = \operatorname{span}\left\{ \left(\frac{1}{5} - \frac{16}{15}x + x^2\right), \left(\frac{1}{4} - x + x^3\right) \right\}$.

Question(3).

$< f_1, f_2 > = \int_0^1 f_1 f_2 \, dx$ is an inner product on P_2.

And since $\mathbb{R}^2 \cong P_2$, then $\mathbb{R}^2 : (a,b) \to P_2 : (a+bx)$.

Hence:
$$< (a_1+b_1x),(a_2+b_2x) > = \int_0^1 (a_1+b_1x)(a_2+b_2x) \, dx$$

$$= \int_0^1 (a_1a_2 + a_1b_2x + a_2b_1x + b_1b_2x^2) \, dx$$

$$= \int_0^1 (a_1a_2 + (a_1b_2 + a_2b_1)x + b_1b_2x^2) \, dx$$

$$= \left[a_1a_2x + \frac{(a_1b_2+a_2b_1)x^2}{2} + \frac{b_1b_2x^3}{3} \right]_0^1$$

$$= a_1a_2 + \frac{a_1b_2+a_2b_1}{2} + \frac{b_1b_2}{3}$$

Since $<(a_1+b_1x),(a_2+b_2x)> = a_1a_2 + \frac{1}{2}(a_1b_2+a_2b_1) + \frac{1}{3} b_1b_2$ is an inner product, then $<(a_1,b_1),(a_2,b_2)> = a_1a_2 + \frac{1}{2}(a_1b_2+a_2b_1) + \frac{1}{3}b_1b_2$ is also an inner product (due to the isomorphic).

Question (4).

Let $Q_1 = (a_1, a_2, a_3, a_4, a_5)$

$Q_2 = (b_1, b_2, b_3, b_4, b_5)$

Now, Cauchy-Schwarz inequality states that

$$\langle Q_1, Q_2 \rangle^2 \leq \langle Q_1, Q_1 \rangle \cdot \langle Q_2 \cdot Q_2 \rangle$$

$$\rightarrow \left[Q_1 \cdot Q_2^T\right]^2 \leq \left(Q_1 \cdot Q_1^T\right)\left(Q_2 \cdot Q_2^T\right)$$

$$\rightarrow \left(\begin{bmatrix} a_1, a_2, a_3, a_4, a_5 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}\right)^2 \leq \left(\begin{bmatrix} a_1, a_2, a_3, a_4, a_5 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}\right)\left(\begin{bmatrix} b_1, b_2, b_3, b_4, b_5 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}\right)$$

$$\rightarrow (a_1 b_1 + a_2 b_2 + a_3 b_3 + a_4 b_4 + a_5 b_5)^2 \leq (a_1 a_1 + \cdots + a_5 a_5)(b_1 b_1 + \cdots + b_5 b_5)$$

$$\therefore (a_1 b_1 + a_2 b_2 + a_3 b_3 + a_4 b_4 + a_5 b_5)^2 \leq (a_1^2 + \cdots + a_5^2)(b_1^2 + \cdots + b_5^2).$$

Question (5)

From Cauchy-Shwarz inequality, we Know that for $u, w \in V$,

we have $\langle u, w \rangle^2 \leq \langle u, u \rangle \langle w, w \rangle$

that is $\langle u, w \rangle^2 \leq \|u\|^2 \|w\|^2$

that is $|\langle u, w \rangle| \leq \|u\| \|w\|$

Now, we have :

$$\|V+W\|^2 = \langle V+W, V+W \rangle$$
$$= \langle V+W, V \rangle + \langle V+W, W \rangle$$
$$= \langle V, V \rangle + \langle W, V \rangle + \langle V, W \rangle + \langle W, W \rangle$$
$$= \langle V, V \rangle + \langle V, W \rangle + \langle V, W \rangle + \langle W, W \rangle$$
$$= \langle V, V \rangle + 2 \langle V, W \rangle + \langle W, W \rangle$$
$$= \|V\|^2 + 2 \langle V, W \rangle + \|W\|^2$$

$\therefore \|V+W\|^2 = \|V\|^2 + 2 \langle V, W \rangle + \|W\|^2$

Since $|\langle u, w \rangle| \leq \|u\| \|w\|$, then

$$\|V+W\|^2 \leq \|V\|^2 + 2\|u\| \|w\| + \|w\|^2$$

$$\|V+W\|^2 \leq (\|V\| + \|W\|)^2$$

$$\|V+W\| \leq \|V\| + \|W\|, \quad \square$$

Question (6)

Orthogonal Basis:

$$D = \text{span}\{\overset{Q_1}{1}, \overset{Q_2}{x^3}, \overset{Q_3}{x^4}\}$$

$$W_1 = Q_1 = 1$$

$$W_2 = Q_2 - \frac{\langle Q_2, W_1\rangle}{\|W_1\|^2} \cdot W_1 = Q_2 - \frac{\langle Q_2, W_1\rangle}{\langle W_1, W_1\rangle}$$

$$= x^3 - \frac{\langle x^3, 1\rangle}{\langle 1, 1\rangle} \cdot (1) = x^3 - \frac{\int_0^1 x^3\,dx}{\int_0^1 1\,dx} \cdot (1)$$

$$= x^3 - \frac{\left[\frac{x^4}{4}\right]_0^1}{[x]_0^1} \cdot (1) = x^3 - \frac{\frac{1}{4}}{1} \cdot (1) = x^3 - \frac{1}{4}.$$

$$W_3 = Q_3 - \frac{\langle Q_3, W_2\rangle}{\|W_2\|^2} \cdot W_2 - \frac{\langle Q_3, W_1\rangle}{\|W_1\|^2} \cdot W_1$$

$$= Q_3 - \frac{\langle Q_3, W_2\rangle}{\langle W_2, W_2\rangle} \cdot W_2 - \frac{\langle Q_3, W_1\rangle}{\langle W_1, W_1\rangle} \cdot W_1$$

$$= x^4 - \frac{\langle x^4, x^3 - \frac{1}{4}\rangle}{\langle x^3 - \frac{1}{4}, x^3 - \frac{1}{4}\rangle} \cdot \left(x^3 - \frac{1}{4}\right) - \frac{\langle x^4, 1\rangle}{\langle 1, 1\rangle} \cdot (1)$$

$$= x^4 - \frac{\int_0^1 x^4\left(x^3 - \frac{1}{4}\right)dx}{\int_0^1 \left(x^3 - \frac{1}{4}\right)^2 dx} \cdot \left(x^3 - \frac{1}{4}\right) - \frac{\int_0^1 x^4\,dx}{\int_0^1 1\,dx} \cdot (1)$$

$$= x^4 - \frac{\left[\frac{x^8}{8} - \frac{x^5}{20}\right]_0^1}{\left[\frac{x}{16} + \frac{x^7}{7} - \frac{x^4}{8}\right]_0^1} \cdot \left(x^3 - \frac{1}{4}\right) - \frac{\left[\frac{x^5}{5}\right]_0^1}{[x]_0^1} \cdot (1)$$

$$= x^4 - \frac{3/40}{9/112} \cdot \left(x^3 - \frac{1}{4}\right) - \frac{1/5}{1} \cdot (1)$$

$$= x^4 - \frac{14}{15}\left(x^3 - \frac{1}{4}\right) - \frac{1}{5}$$

$$= x^4 - \frac{14}{15}x^3 + \frac{14}{60} - \frac{1}{5}$$

$$= x^4 - \frac{14}{15}x^3 + \frac{1}{30}$$

Orthonormal Basis:

$$F_1 = \frac{W_1}{\|W_1\|} = \frac{W_1}{\sqrt{\langle W_1, W_1\rangle}}$$

- $\|W_1\| = \sqrt{\langle W_1, W_1\rangle} = \sqrt{1} = 1$

$$F_1 = \frac{1}{1} = 1$$

$$F_2 = \frac{W_2}{\|W_2\|} = \frac{W_2}{\sqrt{\langle W_2, W_2\rangle}}$$

- $\|W_2\| = \sqrt{\langle W_2, W_2\rangle} = \sqrt{9/112} = \frac{3\sqrt{7}}{28}$

$$F_2 = \frac{x^3 - \frac{1}{4}}{\frac{3\sqrt{7}}{28}} = \frac{4\sqrt{7}}{3}x^3 - \frac{\sqrt{7}}{3}$$

$$F_3 = \frac{W_3}{\|W_3\|} = \frac{W_3}{\sqrt{\langle W_3, W_3\rangle}}$$

- $\|W_3\| = \sqrt{\langle W_3, W_3\rangle} = \sqrt{1/900} = \frac{1}{30}$

$$F_3 = \frac{x^4 - \frac{14}{15}x^3 + \frac{1}{30}}{\frac{1}{30}} = 30x^4 - 28x^3 + 1$$

Question (7).

$$A = \begin{bmatrix} a & -3 \\ b & c \end{bmatrix}$$

Since A is positive definit, then $b = -3$,

and $ac - b^2 > 0$, then $ac - 9 > 0$

and we know $a, c > 0$

hence $ac > 9$

Question (8)

Solve $\langle 4x^3, 1-2x \rangle = \langle C_1(1-2x) + C_2V_2 + C_3V_3 + C_4V_4, 1-2x \rangle$

- $\langle 4x^3, 1-2x \rangle = \int_0^1 4x^3(1-2x)\,dx$

$$= \int_0^1 (4x^3 - 8x^4)\,dx$$

$$= \left[\frac{4x^4}{4} - \frac{8x^5}{5}\right]_0^1$$

$$= \left[x^4 - \frac{8}{5}x^5\right]_0^1$$

$$= 1 - \frac{8}{5}$$

$$= \frac{-3}{5}$$

- $\langle c_1(1-2x) + c_2V_2 + c_3V_3 + c_4V_4 \,,\, 1-2x \rangle$

$$= \langle c_1(1-2x), 1-2x \rangle + \langle c_2V_2, 1-2x \rangle + \langle c_3V_3, 1-2x \rangle + \langle c_4V_4, 1-2x \rangle$$

$$= c_1 \langle 1-2x, 1-2x \rangle + c_2 \cancel{\langle V_2, 1-2x \rangle} + c_3 \cancel{\langle V_3, 1-2x \rangle} + c_4 \cancel{\langle V_4, 1-2x \rangle}$$

Since $1-2x$ and V_2, V_3, V_4 are orthogonal, then their inner product is equal to zero.

$$= c_1 \langle 1-2x, 1-2x \rangle$$

$$= c_1 \int_0^1 (1-2x)^2 \, dx$$

$$= c_1 \int_0^1 (1-4x+4x^2) \, dx$$

$$= c_1 \left[x - \frac{4x^2}{2} + \frac{4x^3}{3} \right]_0^1$$

$$= c_1 \left[1 - \frac{4}{2} + \frac{4}{3} \right]$$

$$= \frac{1}{3} c_1$$

So, we have :

$$-\frac{3}{5} = \frac{1}{3} c_1$$

$$c_1 = \frac{-9}{5}$$

2.14 HW VII

Name————————————, ID ——————

HW 7, MTH 512 , Fall 2019

Ayman Badawi

QUESTION 1. A matrix A, $m \times m$, is nilpotent if $A^n = 0$ for some positive integer n. Let A be a nilpotent matrix 7×7 such that $m_A(\alpha) = \alpha^3$ and $IN(E_0(A)) = 3$. Find all possible Jordan forms of A.

Find $C_A(\alpha)$.

QUESTION 2. Consider the normal dot product on R^n. Let A be a symmetric matrix over R. Convince me that all eigenvalues of A are real[Hint: Define $T : R^n \to R^n$ such that for every $Q = (a_1, a_2, ..., a_n) \in R^n$, $T(Q) = AQ^T$. What is T^*? and use similar argument as in class]

QUESTION 3. Consider the normal dot product on R^n. Let A be an orthogonal (unitary) matrix (i.e, $A^T = A^{-1}$) over R. Convince me that if $\alpha \in C$ is an eigenvalue of A, then $|\alpha| = 1$.[Hint: Define $T : R^n \to R^n$ such that for every $Q = (a_1, a_2, ..., a_n) \in R^n$, $T(Q) = AQ^T$. What is T^*? and use similar argument as in class]

QUESTION 4. Consider the normal dot product on R^n. Let A be a matrix (of course $n \times n$) such that A is nonsingular (i.e., invertible) and $A^T = A$ over R. Let $B = A^2$. Convince me that $B^T = B$, B is invertible, and all eigenvalues of B are real and each eigenvalue is strictly larger than 0 (i.e., B is positive definite, so now you know how to construct positive definite matrices for every $n \times n$ matrix). [Hint: Define $T : R^n \to R^n$ such that for every $Q = (a_1, a_2, ..., a_n) \in R^n$, $T(Q) = AQ^T$ and note that $< T^2(v), v >=< T(v), T(v)$? why?]

QUESTION 5. Given that A, $n \times n$ and the Jordan form of A is $J = J_2^{(3)} \oplus J_2^{(1)} \oplus J_2^{(1)} \oplus J_6^{(3)} \oplus J_6^{(3)}$. Find the value of n, $m_A(\alpha)$, $C_A(\alpha)$, $IN(E_2(A))$, and $IN(E_6(A))$. (note IN(something) means dim(something)). Is A diagnolizable? why?

QUESTION 6. Given a matrix A, 5×5, with $C_A(\alpha) = (\alpha - 3)^3(\alpha + 4)^2$ and $m_A(\alpha) = (\alpha - 3)(\alpha + 4)^2$. Find the JORDAN form of A. For each eigenvalue a of A find $IN(E_a(A))$ (i.e., find $dim(E_a(A))$.

QUESTION 7. Consider the normal dot product on R^n. Let A be a matrix (of course $n \times n$) such that $A^T = A$ over R. Assume that for some nonzero points Q_1 and Q_2 in R^n, we have $AQ_1^T = aQ_1^T$ and $AQ_2^T = bQ_2^T$ for some real numbers a, b such that $a \neq b$.Convince me that Q_1 and Q_2 are orthogonal. [Hint: use some hints from above!]

QUESTION 8. Give me an example of a matrix A such that $C_A(\alpha) = m_A(\alpha) = (\alpha - 1)^4(\alpha + 5)^5$. For the matrix A that you constructed, for each eigenvalue a of A find $IN(E_a(A))$ (i.e., find $dim(E_a(A))$.

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2.15 Solution to HW VII

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MTH 512
HW # 7

Question # 1

$m_A(x) = x^3$ & $IN(E_0) = 3$

$C_A(x) = x^7$

Hence $A_{7\times 7}$ has only two possible Jordan Forms:

$$J = J_0^{(3)} \oplus J_0^{(3)} \oplus J_0^{(1)}$$

$$J = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$J = J_0^{(3)} \oplus J_0^{(2)} \oplus J_0^{(2)}$$

$$J = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Question #2

let A be symmetric matrix $(A = A^T)$, then define

$T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ s.t. $\forall\, Q = (a_1, \dots, a_n) \in \mathbb{R}^n$

$T(Q) = AQ^T$

Now, $\langle T(Q), Q \rangle = \langle AQ^T, Q \rangle$

$$= (AQ^T)^T \cdot Q^T$$

$$= Q A^T \cdot Q^T$$

$$= \langle Q, A^T Q^T \rangle$$

$$= \langle Q, T^*(Q) \rangle$$

Hence, by inner product property $\Rightarrow \langle T(Q), Q \rangle = \langle T^*(Q), Q \rangle$ (*)

$\Rightarrow T(Q) = T^*(Q)$

$\Rightarrow AQ^T = A^T Q^T$

$\Rightarrow T$ is symmetric.

We know that : $T(Q) = \alpha Q$, for $Q \neq 0$

$\langle T(Q), Q \rangle = \langle \alpha Q, Q \rangle = \alpha \langle Q, Q \rangle$

(*) $\langle Q, T^*(Q) \rangle = \langle Q, T(Q) \rangle = \langle Q, \alpha Q \rangle = \bar{\alpha} \langle Q, Q \rangle$

$\Rightarrow \alpha = \bar{\alpha} \Rightarrow \alpha$ is a real number.

Question #3

let A be orthogonal matrix $(A^T = A^{-1})$, then define

$T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ s.t. $\forall\, Q = (a_1, \dots, a_n) \in \mathbb{R}^n$

$T(Q) = AQ^T$

Now, $\langle T(Q), Q \rangle = \langle AQ^T, Q \rangle$

$= (AQ^T)^T \cdot Q^T$

$= Q A^T Q^T$

$= \langle Q, A^T Q^T \rangle$

$= \langle Q, A^{-1} Q^T \rangle$

$= \langle Q, T^*(Q) \rangle$

Hence, $T^*(Q) = A^{-1} Q^T = T^{-1}(Q)$

$\Rightarrow T$ is orthogonal

By : $T(Q) = \alpha Q$, $Q \neq 0$

$\langle T(Q), Q \rangle = \langle \alpha Q, Q \rangle = \alpha \langle Q, Q \rangle$

$\langle Q, T^*(Q) \rangle = \langle Q, T^{-1}(Q) \rangle = \langle Q, \frac{1}{\alpha} Q \rangle = \frac{1}{\bar{\alpha}} \langle Q, Q \rangle$

$\Rightarrow \alpha = \frac{1}{\bar{\alpha}} \quad \Rightarrow \alpha \bar{\alpha} = 1$

$\Rightarrow |\alpha| = 1$

Question # 4

- let $A_{n\times n}$ be invertible matrix s.t. $A^T = A$
- let $B = A^2$

Define $T : \mathbb{R}^n \to \mathbb{R}^n$ s.t. $\forall Q \in \mathbb{R}^n$, $T(Q) = AQ^T$

Define $F : \mathbb{R}^n \to \mathbb{R}^n$ s.t. $\forall P \in \mathbb{R}^n$, $F(P) = A^2 P^T$
$= BP^T$

$\Rightarrow$ want to show that $B^T = B$:

$$B^T = (A^2)^T = (AA)^T = A^T A \quad (\text{since } A^T = A)$$
$$= A^T A^T$$
$$= AA = A^2 = B$$

$\Rightarrow$ want to show that All eigenvalue of B are real :

(same method as Q # 2)

$$\langle F(P), P \rangle = \langle BP^T, P \rangle$$
$$= (BP^T)^T P^T$$
$$= P B^T P^T$$
$$= \langle P, B^T P^T \rangle$$
$$= \langle P, F^*(P) \rangle$$

Hence $F^*(P) = F(P)$

From Q # 2 $\Rightarrow$ all eigenvalues of B are real.

$\Rightarrow$ Want to show that each eigenvalue is > 0 :

$$\langle F(P), P \rangle = \langle T^2(P), P \rangle$$
$$= \langle T(P), T^*(P) \rangle$$
$$= \langle T(P), T(P) \rangle \quad \neq 0$$
$$= \| T(P) \|^2 > 0$$

Since $|A| \neq 0 \Rightarrow \alpha = 0$ is not eigenvalue.

$\Rightarrow \langle F(P), P \rangle > 0$ & $F = F^*$

$\Rightarrow F$ is positive definite.

Question N° 5

$$J = J_2^{(3)} \oplus J_2^{(1)} \oplus J_2^{(1)} \oplus J_6^{(3)} \oplus J_6^{(3)}$$

① $n = 11$

② $m_A(x) = (x-2)^3 (x-6)^3$

③ $C_A(x) = (x-2)^5 (x-6)^6$

④ $IN(E_2(A)) = 3$

⑤ $IN(E_6(A)) = 2$

A is not diagonizable since we have repeated roots.

Question N° 6

$C_A(x) = (x-3)^3 (x+4)^2$

$m_A(x) = (x-3)(x+4)^2$

$\Rightarrow J = J_3^{(1)} \oplus J_3^{(1)} \oplus J_3^{(1)} \oplus J_{-4}^{(2)}$

$\Rightarrow IN(E_3) = 3$ & $IN(E_{-4}) = 1$

$$J = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 0 & -4 \end{bmatrix}$$

$\boxed{\text{Question } \#7}$

let $A_{n\times n}$ be symmetric s.t. $A^T = A$, & Assume non-zero points $Q_1, Q_2 \in \mathbb{R}^n$ & Some real numbers $a \neq b$

s.t. $AQ_1^T = aQ_1^T$

$AQ_2^T = bQ_2^T$

$\Rightarrow$ Show that Q_1 & Q_2 are orthogonal.

$\boxed{\text{Case 1}}$ If $\langle Q_1, Q_2 \rangle = 0$, done

$\boxed{\text{Case 2}}$ If $\langle Q_1, Q_2 \rangle \neq 0$

$$
\begin{aligned}
a\langle Q_1, Q_2\rangle &= \langle aQ_1, Q_2\rangle \\
&= \langle T(Q_1), Q_2\rangle \\
&= \langle Q_1, T^*(Q_2)\rangle \\
&= \langle Q_1, T(Q_2)\rangle \\
&= \langle Q_1, bQ_2\rangle \\
&= \bar{b}\langle Q_1, Q_2\rangle = b\langle Q_1, Q_2\rangle
\end{aligned}
$$

$\Rightarrow$ Hence $a = b$!! contradiction

Since we assume $a \neq b$

$\Rightarrow$ Thus Q_1, Q_2 orthogonal.

Question #8

$$c_A(\alpha) = m_A(\alpha) = (\alpha-1)^4(\alpha+5)^5$$

$\Rightarrow A_{9\times9}$ matrix

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -5 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5 \end{bmatrix} = J$$

$\Rightarrow IN(E_1) = 1$

$IN(E_{-5}) = 1$

2.16 HW VIII

HW 8, MTH 512 , Fall 2019

Ayman Badawi

QUESTION 1. Let A be a skew-symmetric matrix (i.e., $A^T = -A$) ,2019 × 2019. Convince me that A is not invertible.

QUESTION 2. Let $A = J_3^{(2)} \oplus J_2^{(2)} \oplus J_3^{(1)}$. Find the rational form of A.

QUESTION 3. Let $T : V \to V$ be a linear transformation. Consider the linear transformation $F = T^2 + 5T + 2019I : V \to V$. Let $W = Z(F)$. Convince me that $T(w) \in W$ for every $w \in W$.

QUESTION 4. Find the Smith form of $\begin{bmatrix} 3 & 6 & 3 \\ -3 & 0 & 3 \\ -3 & -6 & 0 \end{bmatrix}$ (i.e., find D, R, C such that $D = RAC$ (see class notes))

QUESTION 5. Let $A = \begin{bmatrix} 2 & 4 & 4 & 2 & 6 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$

1) Find $C_A(x)$
2) Use your favorite software and find $m_A(x)$.
3) For each eigenvalue a of A find $IN(E_a)$ (i.e., Find the dimension of the eigenspace of A that corresponds to the eigenvalue a).
4) Find the Jordan form of A
5) Find the rational form of A.

QUESTION 6. 1) First show that $m_A(x) = m_{A^T}(x)$ of course A is $n \times n$(so EASY).

2) Assume A, B, C are $n \times n$ matrices such that A is similar to B and B is similar to C (Recall that M, N are similar iff there exists an invertible matrix Q such that $M = QNQ^{-1}$). Convince me that A is similar to C.

3) Now BIG result Show that if A is an $n \times n$ matrix. Then A is similar to A^T (waw waw result) [Hint: We know that $C_A(x) = C_{A^T}(x)$. By (1) we know $m_A(x) = m_{A^T}(x)$. We know $IN(E_a)$ when a is an eigenvalue of A equals to $IN(E_a)$ when a (same a) as an eigenvalue of A^T (not matter if a is real or complex number). Now what can we say about the rational form of A and A^T ? then use (2), just a beautiful result with easy proof]

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2.17 Solution to HW VIII

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Homework 8

* Question 1: Let A be a skew-symmetric matrix ($A^T = -A$) such that A is 2019×2019

. since A is of odd size, its characteristic polynomial is of odd degree, such a polynomial has at least one real root, hence A has at least one real eigenvalue

Let α be the real eigenvalue of A, then $AX = \alpha X$ for $X \neq 0$.

$$
\begin{aligned}
\text{So } \alpha \langle X, X \rangle &= \langle X, \bar{\alpha} X \rangle \quad \rightarrow \text{ since } \alpha \text{ is real} \\
&= \langle X, \alpha X \rangle \\
&= \langle x, Ax \rangle \\
&= X^T A X \\
&= (A^T x)^T x \\
&= \langle A^T x, x \rangle \\
&= -\langle AX, x \rangle \\
&= -\langle \alpha X, X \rangle \\
&= -\alpha \langle X, X \rangle
\end{aligned}
$$

since $\langle X, X \rangle \neq 0$ then $\alpha = -\alpha \Rightarrow 2\alpha = 0 \Rightarrow \alpha = 0$

thus A is not invertible

* Question 2:

Let $A = J_3^{(2)} \oplus J_2^{(2)} \oplus J_3^{(1)}$

then $C_A(x) = (x-3)^3 (x-2)^2$

and $m_A(x) = (x-3)^2 (x-2)^2$

thus $R_A = C(f_1) \oplus C(f_2) \oplus C(f_3)$

such that $f_1 = (x-3)^2 = x^2 - 6x + 9$

$f_2 = (x-2)^2 = x^2 - 4x + 4$

$f_3 = x - 3$

hence $R_A = \begin{bmatrix} 0 & -9 & 0 & 0 & 0 \\ 1 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$

* Question 3: Let $T : V \to V$ be a linear tranformation and $F : V \to V$ such that $F = T^2 + 5T + 2019I$

. Let $W = Z(F)$

For every $w \in W$ we have:

$$F(w) = 0$$

hence $T(T^2(w) + 5T(w) + 2019w) = T(0)$

$$\Rightarrow T(T^2(w)) + 5T(T(w)) + 2019T(w) = T(0)$$

$$\Rightarrow T^2(T(w)) + 5T(T(w)) + 2019T(w) = 0$$

$$\Rightarrow F(T(w)) = 0$$

$$\Rightarrow T(w) \in Z(F)$$

$\Rightarrow T(w) \in W$ for every $w \in W$

* Question 4:

Let $A = \begin{bmatrix} 3 & 6 & 3 \\ -3 & 0 & 3 \\ -3 & -6 & 0 \end{bmatrix}$

step 1: gcd (all enteries of A) $= d_1 = 3$

step 2: $|A| = |D| = -6 \begin{vmatrix} -3 & 3 \\ -3 & 0 \end{vmatrix} + 6 \begin{vmatrix} 3 & 3 \\ -3 & 3 \end{vmatrix}$

$$= -6(+9) + 6(9+9)$$

$$= -54 + 108$$

$$= 54$$

thus $d_1 = d_2 = 3$ and $d_3 = 6$

Hence $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$

and $D = RAC$

$\longrightarrow$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 3 & 6 & 3 \\ -3 & 0 & 3 \\ -3 & -6 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_1 + R_2 \to R_2$
$R_1 + R_3 \to R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 3 & 6 & 3 \\ 0 & 6 & 6 \\ 0 & 0 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$-2C_1 + C_2 \to C_2$
$-C_1 + C_3 \to C_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 6 \\ 0 & 0 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 6 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 \leftrightarrow C_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 6 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 & -1 & -2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$-C_3 + C_2 \to C_2$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}}_{= R} \qquad \underbrace{\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}}_{= D} \qquad \underbrace{\begin{bmatrix} 1 & 1 & -2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{= C}$$

and $D = RAC$

* Question 5: Let $A = \begin{bmatrix} 2 & 4 & 4 & 2 & 6 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$

1) $C_A(x) = (x-2)^3 (x-3)^2$

2) By using online calculator: $m_A(x) = (x-2)^2 (x-3)^2$

3) We have two eigenvalues 2 and 3:

$\dim(E_2) = 2$

$\dim(E_3) = 1$

4) $J = J_2^{(2)} \oplus J_3^{(2)} \oplus J_2^{(1)}$

$$= \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

5) $R_A = C(f_1) \oplus C(f_2) \oplus C(f_3)$

where $f_1 = (x-2)^2 = x^2 - 4x + 4$

$f_2 = (x-3)^2 = x^2 - 6x + 9$

$f_3 = (x-2)$

thus $R_A = \begin{bmatrix} 0 & -4 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -9 & 0 \\ 0 & 0 & 1 & 6 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$

* Question 6:

1) suppose $m_A(X) = X^m + a_{m-1} X^{m-1} + \dots + a_1 X + a_0$

and $m_{A^T}(X) = X^n + b_{n-1} X^{n-1} + \dots + b_1 X + b_0$

. we know that $m_{A^T}(A^T) = 0$

and $m_A(A^T) = (A^T)^m + a_{m-1} (A^T)^{m-1} + \dots + a_1 A^T + a_0 I_m$

$= (A^m + a_{m-1} A^{m-1} + \dots + a_1 A + a_0 I_m)^T$ since $m_A(A) = 0$

$= 0$

thus $m_{A^T}(X)$ divides $m_A(X)$

and we know that $m_A(A) = 0$

and $m_{A^T}(A) = A^n + b_{n-1} A^{n-1} + \dots + b_1 A + b_0 I_n$

$= [(A^T)^n + b_{n-1} (A^T)^{n-1} + \dots + b_1 A^T + b_0 I_n]^T$ since $m_{A^T}(A^T) = 0$

$= 0$

thus $m_A(X)$ divides $m_{A^T}(X)$

therefore they must be equal

2) A is similar to B $\Rightarrow$ $A = QBQ^{-1}$

B is similar to C $\Rightarrow$ $B = PCP^{-1}$

then $A = Q(PCP^{-1})Q^{-1}$

$= QPC(QP)^{-1}$ $\rightarrow$ let $W = QP \Rightarrow W^{-1} = (QP)^{-1}$

$= WCW^{-1}$

therefore A and C are similar.

3) We know that $C_A(x) = C_{A^T}(x)$

we proved in (1) that $m_A(x) = m_{A^T}(x)$

Moreover, $IN(E_a)$ is the same for A and A^T

So we can conclude that $R_A = R_{A^T}$

and we know that A is similar to R_A

and $R_{A^T} = R_A$ is similar to A^T

thus by (2) A is similar to A^T.

2.18 Handout on Jordan and Rational forms

Questions on Last Lecture

Notes:

Why Do we Care about A is similar to B?

1. $C_A(x) = C_B(x)$
2. $m_A(x) = m_B(x)$
3. eigenvalues of A = eigenvalues of B
4. If α is an eigenvalue of A (and hence it is an eigenvalue of B), $\dim(E_\alpha)$ [considering A] $= \dim(E_\alpha)$ [considering B]

$$A = J_3^{(4)} \oplus J_3^{(2)} \oplus J_1^{(5)} \oplus J_1^{(3)}$$

$$m_A(x) = (x-3)^4 (x-1)^5$$

$$C_A(x) = (x-3)^6 (x-1)^8$$

① (Questions and answers on Last Lecture)

Correction: What I called Canonical form, it is known as Companion matrix, so we will stick with this name.

⇒ Q. $A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. Convince me that A is diagnolizable.

A: By staring, A is the companion matrix of $f(x) = x^3 - x = m_A(x) = C_A(x)$. Since $m_A(x) = x^3 - x = x(x-1)(x+1)$, we know that A is diagnolizable.

⇒ Q. $A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$. Convince me that A is not diagnolizable.

A: By staring, A is the companion matrix of $f(x) = x^3 - 2x^2 + x = C_A(x) = m_A(x)$. Since $m_A(x) = x(x-1)^2$, we know by class-Theorem, A is not diagnolizable.

(Doing L.A by staring 2)

⇒ **Q.** Assume A, 3×3, is symmetric. Will it be possible that $c_A(x) = x(x^2+1)$?

A. No, By class notes, all eigenvalues of A are real.

~~Q. Is A =~~

⇒ **Q.** Is $A = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ triangulizable?

A. ~~No~~ yes, by staring, $c_A(x) = m_A(x) = (x^2-1)^2$. we conclude that Since $m_A(x) = (x-1)^2(x+1)^2$, A is triangulizable (class notes).

⇒ **Q.** Is $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ diagnolizable?

Find $\dim(E_2)$ (note $\dim(E_2) = IN(E_2)$)

A. By staring, $A = J_2^{(3)}$. Hence without calculation $\dim(E_2) = 1$.

③ Doing LA by Staring

⇒ **Q.** Given $C_A(x) = (x-2)^3(x-5)$ and $\dim(E_2) = 2$ (i.e., $IN(E_2) = 2$).

Find $m_A(x)$ and Jordan form of A

A. Let us think and stare at $C_A(x)$.

Sind $\dim(E_2) = 2$ and $\dim(E_5) = 1$, we conclude that A is not diagnolizable. Hence $m_A(x) \neq (x-2)(x-5)$. Thus $m_A(x) = (x-2)^2(x-5)$ or $m_A(x) = (x-2)^3(x-5)$.

Suppose $m_A(x) = (x-2)^2(x-5)$. The Jordan-form of A is $\cancel{J_2^{(2)} \oplus J_5^{(1)}}$ $J_2^{(2)} \oplus J_2^{(1)} \oplus J_5^{(1)}$.

Suppose $m_A(x) = (x-2)^3(x-5)$. Then Jordan-form $J_2^{(3)} \oplus J_5^{(1)}$, this is impossible, since $\dim(E_2) = 2$

Thus $m_A(x) = (x-2)^2(x-5)$ and the Jordan-form of A is $J_2^{(2)} \oplus J_2^{(1)} \oplus J_5^{(1)} \rightarrow$

$$\begin{array}{cc} \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix} & \begin{array}{l} \to J_2^{(2)} \\ \to J_2^{(1)} \\ \to J_5^{(1)} \end{array} \end{array}$$

④ $\boxed{\text{Doing L.A. by Staring}}$

$\Rightarrow$ **Q.** Given $C_A(x)=(x-1)^3(x-2)$, and $\dim(E_1)=1$. Find ~~m_A~~ $m_A(x)$ and the Jordan form of A.

A. Let us ~~state~~ stare at all possible Jordan-Block[s]. Since each Jordan-block contribute only 1 to the dimension of an eigenspace. Clearly $J_1^{(3)} \oplus J_2^{(1)}$ is the Jordan-form of A (note $\dim(E_1)=1$ and $\dim(E_2)=1$).

Hence $m_A(x)=C_A(x)=(x-1)^3(x-2)$

so A is similar to

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \qquad \begin{matrix} \to J_1^{(3)}. \\ \\ \to J_2^{(1)} \end{matrix}$$

5 $\boxed{\text{Doing Linear Algebra by Staring}}$

Q. Assume $J_3^{(2)} \oplus J_3^{(2)} \oplus J_3^{(1)} \oplus J_2^{(3)}$ is the Jordan ~~f~~ form of a matrix A

Q-A 1) What is the size of A? (Smile and say, clearly ~~8~~ 8x8 by staring at (2)+(2)+(1)+(3)

So A is 8×8.

Q-A 2) Find $C_A(x)$: Clearly $C_A(x) = (x-3)^5 (x-2)^3$.

~~What is~~ !!

3) Find $m_A(x)$.

By staring $m_A(x) = (x-3)^2 (x-2)^3$.

4) Find $\dim(E_3)$ and $\dim(E_2)$.

Answer: $\dim(E_3) = 3$ (note each Jordan block contributes one ~~to~~ the dimension)

$\dim(E_2) = 1$

6

Read

Doing Linear Algebra by Staring

Q. A, $n \times n$, is nilpotent if $A^k = 0$-matrix for some positive integer k.

Clearly if A is nilpotent, then ~~0 eigenvalues of A~~ 0 is the only eigenvalue of A (for if α is an eigenvalue of A, then α^k is an eigenvalue of A^k, but $A^k = 0$-matrix, so 0 is the only eigenvalue of A).

Find $C_A(x)$.

A. $C_A(x) = x^n$.

~~So we learn that~~

(7)

Q 7 Let A be an $n \times n$ matrix and nilpotent. Convince me that $A^n =$ 0-matrix.

A. $C_A(x) = x^n$. By Caley-Hamilton Th we know $C_A(A) = A^n = 0$-matrix.

Q 8. Let A, $n \times n$, be idempotent and non zero-matrix. Convince me that $m_A(x) = x-1$ or $m_A(x) = x(x-1)$

A. A is idempotent $\Rightarrow A^2 = A \Rightarrow$ $A^2 - A = 0$-matrix. So let $f(x) = x^2 - x$. Then $f(A) = A^2 - A = 0$-matrix

Hence $m_A(x) = x$ or $m_A(x) = x-1$ or $m_A(x) = x^2 - x$. Since A is non-zero, $m_A(x) \neq x$. If $A = I_n$, then $m_A(x) = x-1$

If ~~$m_A(x)$~~ $A \neq I_n$, then $m_A(x) = x(x-1)$

Q9. Let A be idempotent, $n \times n$, s.t. $A \neq I_n$ and $A \neq 0$-matrix. ~~Show $C_A(x) = x^2 - x$~~ ~~$m_A(x)$~~ $\Longrightarrow$ Show $m_A(x) = x^2 - x = x(x-1)$.

Doing L.A. by staring (8)

A. By Question 8, $m_A(x) \neq x$, $m_A(x) \neq (x-1)$. Hence $m_A(x) = x(x-1)$.

Q. Convince me that every non-zero idempotent matrix is diagnolizable.

A: If $A = I_n$, then A is diagonal.
If $A \neq I_n$, then $m_A(x) = x^2 - x = x(x-1)$.
Hence by class Result, A is diagnolizable.

Q. Let A be nonzero idempotent matrix s.t. $A \neq I_n$. Convince me that $C_A(x) = x^k (x-1)^\ell$ s.t. $\ell + k = n$

A. Since $m_A(x) = x^2 - x$ and $m_A(x)$ and $C_A(x)$ have the same eigenvalues and $\deg(C_A(x)) = n$, we conclude that $C_A(x) = x^\ell (x-1)^k$ s.t. $k + \ell = n$.

(9)

Q. Assume $A^{5\times 5}$ is a non zero idempotent matrix and $A \neq I_5$. Given $\dim(E_0) = 3$. Find the Jordan-form of A.

A. We know $m_A(x) = x(x-1)$. Hence A is diagnolizable. Since $\dim(E_0) = 3$, $\dim(E_1) = 2$. ~~Then~~

Thus Jordan-form is

$$J_0^{(1)} \oplus J_0^{(1)} \oplus J_0^{(1)} \oplus J_1^{(1)} \oplus J_1^{(1)}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow$$ A is similar to this Jordan-form.

~~Q. Stone~~

(10) Doing L.A. by Staring

Q. Stare at this matrix in Jordan form

$$\begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}.$$

Find $C_A(x)$, $m_A(x)$, $\dim(E_2)$, $\dim(E_5)$

A. By staring, A is similar to

$$J_2^{(4)} \oplus J_5^{(2)}$$

Hence $C_A(x) = (x-2)^4(x-5)^2$.

$m_A(x) = (x-2)^4(x-5)^2$

$\dim(E_2) = 1$, $\dim(E_5) = 1$

(note each Jordan Block contributes $\underline{1}$ to the dimension).

3 Section 5: Two Exams and Final

3.1 Exam One

Name————————————, ID ——————

Review Exam one MTH 512 , Fall 2019

Ayman Badawi

QUESTION 1. Let A be a 3×5 such that $A \overbrace{2R_2} B \overbrace{-R_2 + R_3 \to R_3} D = \begin{bmatrix} 1 & 0 & 2 & -1 & 1 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$

(i) Find the solution set to the system $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 6 \end{bmatrix}$ [Hint: Note that the solution set is a subset of R^5 and think!].

(ii) Find Elementary matrices E_1, E_2 such that $E_1E_2A = D$

(iii) Let $D = \begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}\begin{bmatrix}1 & -1\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}\begin{bmatrix}1 & 0\\-2 & 1\end{bmatrix}\begin{bmatrix}2 & 4\\4 & 2\end{bmatrix}$. Find the matrix D without doing the actual multiplication of these 5 matrices [Stare well and think!]

QUESTION 2. (i) Let A be an $n \times n$ invertible matrix. Convince me (i.e. prove) that if a is an eiginvalue of A, then a^{-1} is an eigenvalue of A^{-1}. Also, convince me that $E_a = E_{a^{-1}}$.

(ii) Given A is a 3×3 diagnolizable matrix with eigenvalues 2, -2 such that $E_{-2} = span\{(1, 2, 3,), (-1, -2, -2)\}$ and $E_2 = span\{(-1, -1, -3)\}$.

a. Find $|A|$ and Trace(A)

b. Find a diagonal matrix D and an invertible matrix Q such that $D = QAQ^{-1}$ (Do not calculate Q^{-1}).

c. Find $C_{A^{-1}}(\alpha)$.

d. Find C_{A^2} and calculate A^2.

(iii) Let A be an $n \times n$ matrix. Suppose that there is a real number r such that the sum of all numbers in each column of A equals r. Convince me that r is an eigenvalue of A.

(iv) Let A be a 13×13 matrix. Convince me that A must have at least one real eigenvalue.

(v) Let A be a $4{\times}4$ matrix and $C_A(\alpha) = (\alpha{-}3)^2(\alpha{-}2)^2$ such that $E_3 = span\{(2,1,1,1)\}$ and $E_2 = span\{-2,1,0,1)\}$.

a. What is the solution set to the system $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 5 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$?

b. Let $F = 5I_4 + 2A^{-1} + 3A$. Give me a nonzero point Q and a real number a such that $FQ^T = aQ^T$.

QUESTION 3. Let $A = \begin{bmatrix} -c_5 & a_2 & a_3 & -2c_1 & a_5 \\ c_3 & b_2 & b_3 & -c_1 & b_5 \\ c_1 & -2 & c_3 & -1 & c_5 \end{bmatrix}$. Given A is row-equivalent to $B = \begin{bmatrix} 2 & 4 & 4 & 2 & 4 \\ 0 & 1 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

(a)Find the matrix A.

(b) Find a basis of $Col(A)$.

QUESTION 4. Given $B = \{(0,1,1),(1,0,-1),(2,-2,-1)\}$ is a basis for R^3 and $Q = (2,6,-1) \in R^3$. Find $[Q]_B$.

QUESTION 5. Let $D = \{(3a+5b+2,\ -2b+1,\ 6a+8b+5,\ 6b-3,\ 3a+3b+3) \mid a, b \in R\}$.

(a) Convince me that D is a subspace of R^5.

(b) Find an orthogonal basis of D.

QUESTION 6. Let $A = \begin{bmatrix} 2 & 4 & 1 & -3 \\ -2 & b_2 & b_3 & b_4 \\ -2 & -4 & c_3 & c_4 \\ -2 & -4 & -1 & d_4 \end{bmatrix}$. Assume that a point $Q = (x_1, x_2, x_3, x_4)$ is selected randomly from R^4. Find all possible values of $b_2, b_3, b_4, c_3, c_4, d_4$ so that the system $A \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = Q^T$ has a unique solution.

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3.2 Solution to Exam I

Name————————————, ID ————————

Review Exam one MTH 512 , Fall 2019

Ayman Badawi

QUESTION 1. Let A be a 3×5 such that $A \ \overbrace{2R_2}\ B \ \overbrace{-R_2 + R_3 \to R_3}\ D = \begin{bmatrix} 1 & 0 & 2 & -1 & 1 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$

(i) Find the solution set to the system $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 6 \end{bmatrix}$ [Hint: Note that the solution set is a subset of R^5 and think!].

SOLUTION 1.1. We need to form the augmented matrix. Note that A is the coefficient matrix. Hence $[A| \begin{bmatrix} -1 \\ 1 \\ 6 \end{bmatrix}]$ is the augmented matrix. By hypothesis A is reduced to D by row operations. Hence here we go $[A| \begin{bmatrix} -1 \\ 1 \\ 6 \end{bmatrix}] \ \overbrace{2R_2}\ [B| \begin{bmatrix} -1 \\ 2 \\ 6 \end{bmatrix}] \ \overbrace{-R_2 + R_3 \to R_3}\ D = \begin{bmatrix} 1 & 0 & 2 & -1 & 1 & | & -1 \\ 0 & 1 & 2 & 0 & 3 & | & 2 \\ 0 & 0 & 0 & 1 & 2 & | & 4 \end{bmatrix} \ \overbrace{R_3 + R_1 \to R_1}\ F = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 & | & 3 \\ 0 & 1 & 2 & 0 & 3 & | & 2 \\ 0 & 0 & 0 & 1 & 2 & | & 4 \end{bmatrix}$. Hence we stop and read

$x_1 = 3 - 2x_3 - x_5, x_2 = 2 - 2x_3 - 3x_5, x_4 = 4 - 2x_5$. Note x_1, x_2, x_4 are leading variables and $x_3, x_5 \in R$ (free variables).

Thus the solution set = $\{(3 - 2x_3 - x_5,\ 2 - 2x_3 - 3x_5,\ x_3,\ 4 - 2x_5, x_5) \mid x_3, x_5 \in R\}$

Since the system is not homogeneous, the solution set is a SUBSET of R^5 but NEVER a subspace of R^5 and hence it cannot be written as span. Also; note that we cannot talk about independent number (dimension) [since it is not a Subspace].

(ii) Find Elementary matrices E_1, E_2 such that $E_1 E_2 A = D$

SOLUTION 1.2. By staring at the row operations from A to D and $E_1 E_2 = D$, we see that the first row operation corresponds to E_2 and the second row operation corresponds to E_1. Hence $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

(iii) Let $D = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$. Find the matrix D without doing the actual multiplication of these 5 matrices [Stare well and think!]

SOLUTION 1.3. By staring, we observe that the first 4 matrices are elementary matrices. Hence

$$\begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix} \ \overbrace{-2R_1 + R_2 \to R_2}\ \begin{bmatrix} 2 & 4 \\ 0 & -6 \end{bmatrix} \ \overbrace{2R_2}\ \begin{bmatrix} 2 & 4 \\ 0 & -12 \end{bmatrix} \ \overbrace{-R_2 + R_1 \to R_1}$$

$$\begin{bmatrix} 2 & 16 \\ 0 & -12 \end{bmatrix} \ \overbrace{2R_2 + R_1 \to R_1}\ \begin{bmatrix} 2 & -8 \\ 0 & -12 \end{bmatrix} = D$$

QUESTION 2. (i) Let A be an $n \times n$ invertible matrix. Convince me (i.e. prove) that if a is an eiginvalue of A, then a^{-1} is an eigenvalue of A^{-1}. Also, convince me that $E_a = E_{a^{-1}}$.

SOLUTION 2.1. Since a is an eigenvalue of A and A is invertible, we conclude that $a \neq 0$ and there exists a nonzero point Q in R^n such that $AQ^T = aQ^T$. Multiply both sides with A^{-1}, we get $Q^T = aA^{-1}Q$. Thus $A^{-1}Q^T = \frac{1}{a}Q^T$. Thus $1/a$ is an eigenvalue of A^{-1}.

As we learned from Elementary Math, to show that two sets , say F, K, are equal, we need to show that $F \subseteq K$ and $K \subseteq F$.

Hence we need to show that $E_a \subseteq E_{a^{-1}}$ and $E_{a^{-1}} \subseteq E_a$.

So, let $Q \in E_a$. We show $Q \in E_{a^{-1}}$. Thus $AQ^T = aQ^T$. Multiply both sides with A^{-1}, we get $Q^T = aA^{-1}Q$. Thus $A^{-1}Q^T = \frac{1}{a}Q^T$. Thus $Q \in E_{a^{-1}}$. Hence $E_a \subseteq E_{a^{-1}}$.

Now let $W \in E_{a^{-1}}$. We show $W \in E_a$. Hence $A^{-1}W^T = \frac{1}{a}W^T$. Multiply both sides with A. Thus $W^T = \frac{1}{a}AW^T$. Hence $AW^T = aW^T$. Hence $W \in E_a$, and therefore $E_{a^{-1}} \subseteq E_a$. Since $E_a \subseteq E_{a^{-1}}$ and $E_{a^{-1}} \subseteq E_a$, we conclude that $E_{a^{-1}} = E_a$.

(ii) Given A is a 3×3 diagnolizable matrix with eigenvalues 2, -2 such that $E_{-2} = span\{(1,2,3),(-1,-2,-2)\}$ and $E_2 = span\{(-1,-1,-3)\}$.

a. Find $|A|$ and Trace(A)

SOLUTION 2.2. Since A is diagnolizable, by staring at E_{-2} and E_2 we conclude that 2 is repeated once and -2 is repeated twice. Hence |A| = (-2)(-2)(2) = 8. Trace(A) = -2 + -2 + 2 = -2.
NOTE that A is diagnolizable is not needed in this question! right?

b. Find a diagonal matrix D and an invertible matrix Q such that $D = QAQ^{-1}$ (Do not calculate Q^{-1}).

SOLUTION 2.3. As explained in class, many possibilities. For example: $D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$, $Q = \begin{bmatrix} 1 & -1 & -1 \\ 2 & -1 & -2 \\ 3 & -3 & -2 \end{bmatrix}$

c. Find $C_{A^{-1}}(\alpha)$.

SOLUTION 2.4. From question (2), we conclude that $\frac{-1}{2}, \frac{-1}{2}, \frac{1}{2}$ are the eigenvalues of A^{-1}. Hence $C_{A^{-1}}(\alpha) = (\alpha + \frac{1}{2})^2(\alpha - \frac{1}{2})$.

d. Find C_{A^2} and calculate A^2.

SOLUTION 2.5. Let Q, D as in Solution 2.3. Hence $Q^{-1}DQ = A$. Thus $Q^{-1}D^2Q = A^2$. Stare at D^2. You observe that $D^2 = 4I_3$. Hence $4Q^{-1}I_3Q = A^2$. Hence $A^2 = 4I_3$. Thus $C_{A^2}(\alpha) = |\alpha I_3 - 4I_3| = (\alpha - 4)^3$.

(iii) Let A be an $n \times n$ matrix. Suppose that there is a real number r such that the sum of all numbers in each column of A equals r. Convince me that r is an eigenvalue of A.

SOLUTION 2.6. Consider the matrix A^T. Then the sum of all numbers in each row of A^T equals r. Hence $A^T \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = r \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$. Then r is an eigenvalue of A^T. We know that A^T and A have the same eigenvalues. Thus r is an eigenvalue of A.

(iv) Let A be a 13×13 matrix. Convince me that A must have at least one real eigenvalue.

SOLUTION 2.7. Note that the degree of $C_A(\alpha)$ is 13. So we set $C_A(\alpha) = 0$. Common knowledge (public knowledge) every polynomial of odd degree must have at least one real root. Thus A must have at least one real eigenvalue.

(v) Let A be a 4×4 matrix and $C_A(\alpha) = (\alpha-3)^2(\alpha-2)^2$ such that $E_3 = span\{(2,1,1,1)\}$ and $E_2 = span\{(-2,1,0,1)\}$.

a. What is the solution set to the system $A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 5\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$?

SOLUTION 2.8. By staring at $C_A(\alpha)$. We conclude that 5 is not an eigenvalue of A. Hence the solution set is $\{(0,0,0,0)\}$.

b. Let $F = 5I_4 + 2A^{-1} + 3A$. Give me a nonzero point Q and a real number a such that $FQ^T = aQ^T$.

SOLUTION 2.9. Fist observe that A^{-1} exists, since $|A| = (2)(2)(3)(3) = 36 \neq 0$. Choose any nonzero point Q in E_2 or E_3. We Know from solution 2.1 that $Q \in E_{\frac{1}{2}}$ or $Q \in E_{\frac{1}{3}}$ (note $E_{\frac{1}{2}}$ and $E_{\frac{1}{3}}$ are eigenspaces of A^{-1}).
Let us choose $Q = (-2,1,0,1) \in E_2$. Then
$FQ^T = [5I_4 + 2A^{-1} + 3A]Q^T = 5I_4Q^T + 2A^{-1}Q^T + 3AQ^T = 5Q^T + 2(0.5Q^T) + 3(2Q^T) =$
$5Q^T + Q^T + 6Q^T = 12Q^T$ (i.e., 12 is an eigenvalue of F).

QUESTION 3. Let $A = \begin{bmatrix} -c_5 & a_2 & a_3 & -2c_1 & a_5 \\ c_3 & b_2 & b_3 & -c_1 & b_5 \\ c_1 & -2 & c_3 & -1 & c_5 \end{bmatrix}$. Given A is row-equivalent to $B = \begin{bmatrix} 2 & 4 & 4 & 2 & 4 \\ 0 & 1 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.
(a)Find the matrix A.

SOLUTION 3.1. Note A_i means the ith column of A and $_iA$ means the ith row of A
By staring, $Row(A) = span\{(2,4,4,2,4),(0,1,1,3,1)\}$. As explained, each row of A is a linear combination of (2, 4, 4, 2, 4), (0, 1, 1, 3, 1)
Hence $_3A = (c_1,-2,c_3,-1,c_5) = a(2,4,4,2,4) + b(0,1,1,3,1) = (2a, 4a+b, 4a+b, 2a+3b, 4a+b)$. Find a, b. Hence $4a + b = -2$ and $2a + 3b = -1$. Now solve! we get $a = -0.5$ and $b = 0$. Thus $_3A = (-1,-2,-2,-1,-2)$. Hence $c_1 = -1$, $c_3 = -2$, $c_5 = -2$.
Similarly $_2A = (-2,b_2,b_3,1,b_5) = a(2,4,4,2,4) + b(0,1,1,3,1) = (2a, 4a+b, 4a+b, 2a+3b, 4a+b)$. Find a, b. Hence $2a = -2$ and $2a + 3b = 1$. Now solve! we get $a = -1$ and $b = 1$. Thus $_2A = (-2,-3,-3,1,-3)$.
Similarly $_1A = (2,a_2,a_3,2,a_5) = a(2,4,4,2,4) + b(0,1,1,3,1) = (2a, 4a+b, 4a+b, 2a+3b, 4a+b)$. Find a, b. Hence $2a = 2$ and $2a + 3b = 2$. Now solve! we get $a = 1$ and $b = 0$. Thus $_1A = (2,4,4,2,4)$.
Hence $A = \begin{bmatrix} 2 & 4 & 4 & 2 & 4 \\ -2 & -3 & -3 & 1 & -3 \\ -1 & -2 & -2 & -1 & -2 \end{bmatrix}$.

(b) Find a basis of $Col(A)$.
As explained, to find a basis for Col(A). We stare at B, we locate the columns in B that have the "leaders". Here we see that the leaders are located in B_1 and B_2. Thus we MUST choose A_1, A_2 from A to form a basis for $Col(A)$.
Hence a basis for Col(A) is $Badawi = \{(2,-2,-1),(4,-3,-2)\}$.
Hence $Col(A) = span\{(2,-2,-1),(4,-3,-2)\}$.

QUESTION 4. Given $B = \{(0,1,1),(1,0,-1),(2,-2,-1)\}$ is a basis for R^3 and $Q = (2,6,-1) \in R^3$. Find $[Q]_B$.

SOLUTION 4.1. Form a matrix P, 3×3, where each column of P is a point in B. Now you may solve the system $PX = Q^T$. Then the point in the solution set is $[Q]_B$. Another way, find P^{-1}. Then $P^{-1}Q^T = [Q]_B$.

QUESTION 5. Let $D = span\{(3a+5b+2,\ -2b+1,\ 6a+8b+5,\ 6b-3,\ 3a+3b+3) \mid a,b \in R\}$.
(a) Convince me that D is a subspace of R^5.

SOLUTION 5.1. As explained, D will be a subspace "if each coordinate can be written as linear combination of linear variables." There are many ways. For example: Let $w = 3a+5b+2, v = -2b+1$. Note that $w, v \in R$ (since a, b in R). Hence $6a+8b+5 = 2w+v, 6b-3 = -3v, 3a+3b+3 = w+v$.
Thus $D = span\{(w,\ v,\ 2w+v,\ -3v,\ w+v) \mid w,v \in R\}$. Hence $D = span\{(1,0,2,0,1),(0,1,1,-3,1)\}$

(b) Find an orthogonal basis of D.

SOLUTION 5.2. Just Use Gram Schmidt Method.

QUESTION 6. Let $A = \begin{bmatrix} 2 & 4 & 1 & -3 \\ -2 & b_2 & b_3 & b_4 \\ -2 & -4 & c_3 & c_4 \\ -2 & -4 & -1 & d_4 \end{bmatrix}$. Assume that a point $Q = (x_1, x_2, x_3, x_4)$ is selected randomly from R^4. Find all possible values of $b_2, b_3, b_4, c_3, c_4, d_4$ so that the system $A \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = Q^T$ has a unique solution.

SOLUTION 6.1. We know that the claim will be correct iff $|A| \neq 0$. So we set $|A| \neq 0$. So let us calculate $|A|$.

$$A = \begin{bmatrix} 2 & 4 & 1 & -3 \\ -2 & b_2 & b_3 & b_4 \\ -2 & -4 & c_3 & c_4 \\ -2 & -4 & -1 & d_4 \end{bmatrix} \overbrace{R_1 + R_2 \to R_2} \overbrace{R_1 + R_3 \to R_3} \overbrace{R_1 + R_4 \to R_4} B = \begin{bmatrix} 2 & 4 & 1 & -3 \\ 0 & b_2 + 4 & b_3 + 1 & b_4 - 3 \\ 0 & 0 & c_3 + 1 & c_4 - 3 \\ 0 & 0 & 0 & d_4 - 3 \end{bmatrix}.$$

Hence $|A| = |B| = 2(b_2 + 4)(c_3 + 1)(d_4 - 3)$.

Thus $|A| \neq 0$ if $b_2 \neq -4$, $c_3 \neq -1$, $d_4 \neq 3$, $b_3, b_4, c_4 \in R$.

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3.3 Exam Two

Exam TWO, MTH 512 , Fall 2019

Ayman Badawi

QUESTION 1. Let $T : V \to V$ be a linear transformation that is invertible, where V is an inner product vector space over R. Assume that $T^* = T^{-1}$. Convince me that $< T(v), T(w) >=< v, w >$ for every $v, w \in V$.

QUESTION 2. Let T be a linear transformation from a vector space V over R to R such that $T(v_1) = 2$, $T(v_2) = 4$, and $T(v_3) = 7$, where $B = \{v_1, v_2, v_3\}$ is a basis of V. Convince me that there is a UNIQUE point $Q \in R^3$ such that $T(v) =< Q, X >$, where $X = [v]_B$ (the coordinate of v with respect to B), and $<\ ,\ >$ is the normal dot product on R^3.

QUESTION 3. Let $T : P_5 \to R^4$ such that $M_{B,B'} = \begin{bmatrix} 1 & 2 & 4 & 6 & -2 \\ 0 & 2 & 4 & 3 & 5 \\ 0 & 4 & 8 & 6 & 10 \\ 3 & 6 & 12 & 18 & -6 \end{bmatrix}$ be the matrix presentation of T with respect to $B = \{x^4, 1 + x^4, 1 + x + x^4, x^3 + x^4, x^2 + x^4\}$ and $B' = \{(2, 4, 6, 6), (-2, 4, 6, 6), (-2, -4, 6, 6), (-2, -4, -6, 6)\}$.

(i) Find the fake standard matrix presentation of T.

(ii) Find $T(4x^2 + x^4)$. Then find all (describe all) elements in P_5, say v, so that $T(v) = T(4x^2 + x^4)$.

QUESTION 4. Given $B = \{T_1, T_2, T_3, T_4\}$ is a basis for $Hom(P_2, P_2)$, where $T_1 : P_2 \to P_2$ such that $T_1(a_1 + a_2x) = (a_1 + a_2) + a_1x$ and $T_2 : P_2 \to P_2$ such that $T_2(a_1 + a_2x) = (a_1 + a_2)x$. Find T_3 and T_4. (i.e., you must show that T_1, T_2, T_3, T_4 are independent)

QUESTION 5. Let V be an inner product space over R. Convince me that $||v+w||^2 = ||v||^2+||w||^2$ for every orthogonal elements $v, w \in V$.

QUESTION 6. Let $W = span\{A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, K = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\}$ Find a basis for $W^\perp$ (note $< A, B >= Trace(B^T A)$)

QUESTION 7. Let $T : R^4 \to R^4$ be a linear transformation (operator) such that the matrix presentation of T with respect to the basis $B = \{(1, 1, 1, 1), (-1, 1, 1, 1), (-1, -1, 1, 1), (-1, -1, -1, 1)\}$ is $M_B = \begin{bmatrix} 0 & 0 & 0 & -4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

(i) Find $C_T(x)$ and $m_T(x)$.

(ii) Convince me that T is diagnolizable.

(iii) Find the standard matrix presentation of T^2

(iv) Let $F = 5T^2 - T^4 - I$ (then F is an operator from R^4 into R^4). Convince me that 3 is an eigenvalue of F. Find an orthonormal basis of $E_3(F)$.

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3.4 Solution to Exam II

Name———————————————, ID ——————————

Exam TWO, MTH 512 , Fall 2019

Ayman Badawi

QUESTION 1. Let $T : V \to V$ be a linear transformation that is invertible, where V is an inner product vector space over R. Assume that $T^* = T^{-1}$. Convince me that $< T(v), T(w) >=< v, w >$ for every $v, w \in V$.

Proof. Let $v \in V$. Then $< T(v), T(w) >=< v, T^*T(w) >=< v, T^{-1}T(v) >=< v, v >$

QUESTION 2. Let T be a linear transformation from a vector space V over R to R such that $T(v_1) = 2$, $T(v_2) = 4$, and $T(v_3) = 7$, where $B = \{v_1, v_2, v_3\}$ is a basis of V. Convince me that there is a UNIQUE point $Q \in R^3$ such that $T(v) =< Q, X >$, where $X = [v]_B$ (the coordinate of v with respect to B), and $<\ ,\ >$ is the normal dot product on R^3.

Proof. Let $v \in V$. Then $v = av_1 + bv_2 + cv_3$. Hence $[v]_B = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Now $M_B = [2\ \ 4\ \ 7]$ is the matrix presentation of T with respect to B. Hence $T(v) = M_B[v]_B$. Thus let $Q = (2, 4, 7) \in R^3$. Then $T(v) =< Q, [v]_B >$. Now we show that Q is unique. Assume $F = (m, n, d) \in R^3$ such that $T(v) =< F, [v]_B >$. Then $T(v_1) =< F, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} >= m = 2$, $T(v_2) =< F, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} >= n = 4$, and $T(v_3) =< F, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} >= d = 7$. Thus $F = Q$.

QUESTION 3. Let $T : P_5 \to R^4$ such that $M_{B,B'} = \begin{bmatrix} 1 & 2 & 4 & 6 & -2 \\ 0 & 2 & 4 & 3 & 5 \\ 0 & 4 & 8 & 6 & 10 \\ 3 & 6 & 12 & 18 & -6 \end{bmatrix}$ be the matrix presentation of T with respect to $B = \{x^4, 1 + x^4, 1 + x + x^4, x^3 + x^4, x^2 + x^4\}$ and $B' = \{(2, 4, 6, 6), (-2, 4, 6, 6), (-2, -4, 6, 6), (-2, -4, -6, 6)\}$.

(i) Find the fake standard matrix presentation of T.

To find M_f (fake M), we use $\{1, x, x^2, x^3, x^4\}$ as the standard basis of P_5 and $\{e_1, e_2, e_3, e_4\}$ as the standard basis of R^4. Let $Q = \begin{bmatrix} 2 & -2 & -2 & -2 \\ 4 & 4 & -4 & -4 \\ 6 & 6 & 6 & -6 \\ 6 & 6 & 6 & 6 \end{bmatrix}$. Note that x^4 is viewed as $(0, 0, 0, 0, 1)$ in R^5 (since I am using $\{1, x, x^2, x^3, x^4\}$ as the standard basis of P_5, if you use $\{x^4, x^3, x^2, x, 1\}$ as the standard basis of P_5, then x^4 is viewed as (1, 0, 0, 0, 0) in R^5. So let $P = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$.

Hence we know that $M_{B,B'} = Q^{-1}M_fP$. Thus $M_f = QM_{B,B'}P^{-1}$. Now use the available (multiplication, Inverse) software and do the calculation (make sure that you know how to use the software correctly).

(ii) Use (i) and find $Range(T)$ and $Z(T)$.

To find Range(T): Put M_f in the available software, Transform M_f to echelon form, say B. Stare at the columns in B that have the leaders. Here, TWO columns in B will have the leaders. So $IN(Range(T)) = 2$. YOU MUST FIND THE CORRESPONDING TWO COLUMNS in M_f (class notes). Thus $Range(T) = span\{$ The corresponding two columns in $M_f\}$.

To find Z(T): Solve the homogeneous system $M_fX = 0$. Put the system in the available software. The software will not write it as span. From class notes, you know how to write it as span. In this question, the solution set of the homogeneous system = $span\{$3 independent points in $R^5\}$. Note that $Z(F)$ "lives" inside P_5. So translate each point to a polynomial in P_5 (see class notes). Thus $Z(T) = span\{P_1, P_2, P_3\}$.

(iii) Find $T(4x^2 + x^4)$. Then find all (describe all) elements in P_5, say v, so that $T(v) = T(4x^2 + x^4)$.

To find $T(4x^2 + x^4)$. Do this multiplication (using the software) $M_f \begin{bmatrix} 0 \\ 0 \\ 4 \\ 0 \\ 1 \end{bmatrix}$. Done.

This is an application of a question in one of the home works. $T^{-1}(4x^2 + x^4) = \{4x^2 + x^4 + h \mid h \in Z(T)\}$. You already calculated $Z(T)$. Done.

QUESTION 4. Given $B = \{T_1, T_2, T_3, T_4\}$ is a basis for $Hom(P_2, P_2)$, where $T_1 : P_2 \to P_2$ such that $T_1(a_1 + a_2x) = (a_1 + a_2) + a_1x$ and $T_2 : P_2 \to P_2$ such that $T_2(a_1 + a_2x) = (a_1 + a_2)x$. Find T_3 and T_4. (i.e., you must show that T_1, T_2, T_3, T_4 are independent)

All of you got it right. For example let $T_3(a_1 + a_2x) = a_2$, $T_4(a_1 + a_2x) = a_2x$

QUESTION 5. Let V be an inner product space over R. Convince me that $||v+w||^2 = ||v||^2 + ||w||^2$ for every orthogonal elements $v, w \in V$.

$||v + w||^2 =< v + w, v + w >=< v, v > +2 < v, w > + < w, w >= ||v||^2 + ||w||^2$ (since v, w are orthogonal, i.e., $< v, w >= 0$.)

QUESTION 6. Let $W = span\{A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, K = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\}$ Find a basis for $W^\perp$ (note $< A, B >= Trace(B^T A)$)

Let $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Hence $Trace(B^T A) = 0$ and $Trace(B^T K) = 0$. Hence $a + b = 0$ and $a + d = 0$. Solution set to the homogeneous system is $\{(a, -a, c, -a) \mid a, c \in R\}$ = span$\{(1, -1, 0, -1), (0, 0, 1, 0)\}$. Now translate to matrices. Hence $W^\perp = span\{\begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\}$. **[some of you used the fake-dot product on $R^{2\times 2}$, so I accepted that.. but next time I will not]**

QUESTION 7. Let $T : R^4 \to R^4$ be a linear transformation (operator) such that the matrix presentation of T with respect to the basis $B = \{(1, 1, 1, 1), (-1, 1, 1, 1), (-1, -1, 1, 1), (-1, -1, -1, 1)\}$ is $M_B = \begin{bmatrix} 0 & 0 & 0 & -4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

(i) Find $C_T(x)$ and $m_T(x)$. By staring, M_B is the companion matrix of the polynomial $x^4 - 5x^2 + 4$. Hence we know (by class notes) that $C_T(x) = m_T(x) = x^4 - 5x^2 + 4$.

(ii) Convince me that T is diagnolizable. Since $m_T(x) = x^4 - 5x^2 + 4 = (x^2 - 1)(x^2 - 4) = (x-1)(x+1)(x-2)(x+2)$ (i.e., $m_T(x)$ is a product of distinct linear factors), by class notes T is diagnolizable.

(iii) Find the standard matrix presentation of T^2

Two solutions are accepted:

(1) Assume B is the basis for the domain and the co-domain. Hence $P = \begin{bmatrix} 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

We know that $M_B = P^{-1}MP$. Hence $M = PM_BP^{-1}$ is the standard matrix presentation of T. By class notes (old HW), the standard matrix presentation of T^2 is M^2. Use the available software (multiplication, inverse) to find M and M^2.

(2)Assume B is the basis for the domain and the standard basis $\{e_1, e_2, e_3, e_4\}$ is the basis for the co-domain. Hence $P = \begin{bmatrix} 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$, and $Q = I_4$.

We know that $M_B = I_4^{-1}MP$. Hence $M = I_4M_BP^{-1} = M_BP^{-1}$ is the standard matrix presentation of T. By class notes (old HW), the standard matrix presentation of T^2 is M^2. Use the available software (multiplication, inverse) to find M and M^2.

(iv) Let $F = 5T^2 - T^4 - I$ (then F is an operator from R^4 into R^4). Convince me that 3 is an eigenvalue of F. Find an orthonormal basis of $E_3(F)$.

Process of thinking: By staring $5T^2 - T^4 - I$ is some how related to $C_T(x) = x^4 - 5x^2 + 4$ (some of you observed that). We know (class notes) $C_T(T) = T^4 - 5T^2 + 4I = 0$. Thus $3I = 5T^2 - T^4 - I = F$. Hence $3I(v) = F(v) = 3v$ for every $v \in R^4$. Hence 3 is an eigenvalue of F and $E_3(F) = R^4$. Hence an orthonormal basis is $\{e_1, e_2, e_3, e_4\}$. DONE

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3.5 Final Exam

Name——————————————, ID ————————

Final Exam, MTH 512, Fall 2019

Ayman Badawi

Score = $\frac{\quad}{100}$

QUESTION 1. (4 points) Let $T : V \rightarrow V$ be a linear transformation that is invertible, where V is an inner product vector space over R. Assume that $T^* = T^{-1}$. Assume that $T(v), T(w)$ are nonzero orthogonal elements of V for some nonzero elements $v, w \in V$. Convince me that v, w are orthogonal elements in V.

QUESTION 2. (5 points) Let $T : V \rightarrow V$ be a linear transformation where V is a vector spaces over R and $IN(V) = 3$ (i.e., $dim(V) = 3$). Given $M = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 3 & 5 \\ 2 & 0 & 0 \end{bmatrix}$ is the matrix presentation of T with respect to an ordered basis $\{v_1, v_2, v_3\}$. Convince me that T is invertible. Find $T^{-1}(v_3)$. Convince me that $T^2 - 4T + 3I : V \rightarrow V$ is not invertible (singular).

QUESTION 3. (4 points) Let $T : V \rightarrow V$ be a linear transformation. Consider the linear transformation $F = 2T^3 + 4T^2 + 512I : V \rightarrow V$. Let $W = Z(F)(Ker(F))$. Convince me that $T(w) \in W$ for every $w \in W$.

QUESTION 4. Let $T : P_5 \to R^4$ such that $M_{B,B'} = \begin{bmatrix} 1 & 1 & 1 & -1 & -1 \\ -1 & -1 & -1 & 1 & 1 \\ 2 & 2 & 2 & -2 & -2 \\ 3 & 3 & 3 & -3 & -3 \end{bmatrix}$ be the matrix presentation of T with respect to $B = \{x^4, 1+x^4, 1+x+x^4, x^2+x^4, x^3+x^4\}$ and
$B' = \{(1,1,1,1), (-1,1,0,1), (-2,-2,1,1), (-1,-1,-1,0)\}$.

(i) **(4 points)** Find the fake standard matrix presentation of T. (note that the Fake Matrix Presentation of T is with respect to $\{1, x, x^2, x^3, x^4\}$ and $\{e_1, e_2, e_3, e_4\}$). (you may use the available software)

(ii) **(3 points)** Write Range(T) as span of independent points.(you may use the available software)

(iii) **(3 points)** Write $Z(T)(Ker(T))$ as span of some independent polynomials.(you may use the help of the available software)

(iv) **(2 points)** Find $T(5 + 2x - 4x^3)$. Then find $T^{-1}(5 + 2x - 4x^3)$.

QUESTION 5. Let $T : R^3 \to R^3$ such that $T(1, 0, 1) = (1, 1, 1)$, $T(-1, 1, 1) = (-2, -2, -2)$, and $T(-1, 0, 1) \in Z(T)$. Consider the DOT PRODUCT on R^n.

(i) **(4 points)** Find $T^* : R^3 \to R^3$.

(ii) **(2 points)** write Range of T^* as span of some independent points.(you may use the help of the available software)

(iii) **(3 points)** Write $Z(T)$ as span of some independent points.(you may use the help of the available software)

(iv) **(3 points)** Find $(Z(T))^\perp$ (i.e., find the subspace of R^3 that is orthogonal to $Z(T)$).(you may use the help of the available software) Stare at your answer in (ii) and your answer in (iv). Any connection.

QUESTION 6. (5 points) Consider the normal dot product on R^n. Let A be a symmetric matrix over R. Convince me that all eigenvalues of A are real.

QUESTION 7. (5 points) Let $T : V \to V$ be a linear transformation. Assume that $T^2 = T$. Convince me that $Range(T) \cap Z(T) = 0_v$.

QUESTION 8. (4 points) Consider the normal dot product on R^n. Let A be a matrix (of course $n \times n$) such that $A^T = A$ over R. Assume that for some nonzero points V and W in R^n, we have $AV^T = aV^T$ and $AW^T = bW^T$ for some real numbers a, b such that $a \neq b$. Convince me that V and W are orthogonal.

QUESTION 9. (5 points) Consider the normal dot product on R^n. Let A be a matrix (of course $n \times n$) such that A is nonsingular (i.e., invertible) and $A^T = A$ over R. Let $B = A^2$. Convince me that $B^T = B$, B is invertible, and all eigenvalues of B are real and each eigenvalue is strictly larger than 0 (i.e., B is positive definite)

QUESTION 10. Let $J = J_{-1}^{(2)} \oplus J_2^{(2)} \oplus J_{-1} \oplus J_2$ be the Jordan form of a matrix A.

(i) **(3 points)** Find $C_A(x)$

(ii) **(3 points)** Find $m_A(x)$

(iii) **(3 points)** For each eigenvalue a of A find $IN(E_a)$ (i.e., $dim(E_a)$).

(iv) **(3 points)** Find the rational form of A.

(v) **(3 points)** Is A diagnolizable? explain?

QUESTION 11. Let $A = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

(i) **(3 points)**Find $C_A(x)$ (you may use the available software calculators) OR find it by HAND.

(ii) **(4 points)**Find $m_A(x)$ (you may use the available software calculators) OR find it by hand (maybe LONG)

(iii) **(3 points)** Find the Rational Form of A

(iv) **(3 points)** Find the Jordan Form of A

QUESTION 12. (5 points) Let $T : V \to V$ be a linear transformation that is invertible, where V is a finite dimensional inner product vector space over R. Assume that $T^* = -T$. Convince me that

$$C_T(x) = (x^2 + a_1)^{n_1}(x^2 + a_2)^{n_2} \cdots (x^2 + a_m)^{n_m}$$

, where $a_1, a_2, ..., a_m$ are distinct nonzero positive real numbers, and $n_1, ..., n_m$ are positive integers.

QUESTION 13. (5 points) Let $T : R^3 \to R^3$ be a nonzero non-diagnolizable linear transformation. Given $T^3 - 4T^2 + 4T = 0$. Find all Jordan forms of the standard matrix presentation of T. Find all Rational forms of the standard matrix presentation of T.

QUESTION 14. (6 points) $A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$. Find the SMITH form of A over Z (i.e., find invertible matrices R, C over Z and a diagonal matrix D over Z (with special property as explained in class) such that $D = RAC$)

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